Pole placement in a union of regions with prespecified subregion allocation

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Index Terms

Pole placement, LMI, Union of regions, Prespecified allocation.

Abstract

This paper proposes a method to compute a static state feedback control law which achieves a non strict pole assignment. The specification on the closed-loop poles is given in terms of a clustering region. The originality of this work is the choice of the region which can result from the union of disjoint and non symmetric subregions. Such a choice is made possible by a technique that enables a partial pole placement via aggregation. The distribution of poles in various subregions can be chosen. The proposed conditions are formulated through an LMI approach and some robustness aspects are also considered from an analytical point of view.

1 Introduction

When controlling a linear model, pole placement is a well known method to reach some desired transient performances. Indeed, the shape of the transient response is strongly influenced, in terms of settling time and damping ratio, by the location of the poles in Laplace's plane. A **strict** pole placement is always achievable by a state feedback control law provided that the model is controllable [1]. Whether such a placement is performed by a polynomial approach [2, 3, 4] or it is obtained, in a state-space context, through a classical eigenstructure assignment [5]. For the discrete-time case, see [6, 7, 8].

Both approaches lead to a **strict** placement. However, a slight migration of the poles may not induce a strong modification of the transient behaviour. For this reason, many contributions dealt with **non strict** placement in a region of the complex plane, for instance a sector [9, 10] or a disc [11, 12, 13]. To reach such a goal, Linear Matrix Inequalities (LMI) have revealed as very useful [14].

Nevertheless, the classical regions are usually connected and often even convex. As some sort of compromise between **strict** and **non strict** placement, this contribution aims at assigning the eigenvalues of the closed-loop state matrix in a union of several possibly disjoint and non symmetric subregions. It is of particular interest in the control of systems with separate dynamics (fast ones and slow ones) or when a specified damping ratio is desired. There are few (maybe no ?) techniques to perform such a **non strict** placement to the authors'knowledge. This paper is an attempt to remedy this lack of techniques.

The paper is organized as follows: after this introduction, the section 2 proposes a method for **non strict** partial pole placement by static state feedback thanks to an aggregation technique. The conditions are expressed in terms of LMI. In the section 3, this partial pole placement method is used to achieve a **non strict** but complete pole placement in a union of several subregions. The distribution of the closed closed-loop spectrum in the various subregions can be chosen. In the section 4, an *a posteriori* analysis of the robustness of this pole placement against a norm-bounded uncertainty is made with the help of existing tools. Section 5 is dedicated to an illustration before concluding in section 6.

Notations

M^* denotes the conjugate transpose of matrix M and $||M||_2$ stands for 2-norm of matrix M induced from the Euclidean vector-norm *i. e.* the maximum singular value. M^{-1} is the inverse of square matrix M and $\otimes$ denotes the matrix Kronecker product. M^H is the Hermitian matrix defined by $M+M^*$, $\mathbf{0}$ and $\mathbf{I}$ denote a zero matrix and an identity matrix of appropriate dimension respectively. Matrix inequalities are considered in the sense of Löwner *i. e.* <0 (resp. ≤ 0) stands for "(semi-)negative definite" and >0 (resp ≥ 0) stands for "(semi-)positive definite". Finally, $\mathbf{i}$ denotes the imaginary unit.

2 Partial pole placement

In this section, an aggregation technique is used to propose a very simple method to compute a static state feedback control law which performs a partial pole placement.

Let the system be described by

$$\delta x(t) \quad = \quad Ax(t) + Bu(t)$$

where δ is an operator defined as

$$\delta x(t) = \begin{cases} x(t+1) & \text{for discrete-time systems} \\ \dot{x}(t) & \text{for continuous-time systems} \end{cases}$$

Assume that $(A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m})$ is a controllable pair and that the control law aims at assigning p new closed-loop eigenvalues.

2.1 Aggregation technique

Let C be a matrix defined as

$$C = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \end{bmatrix} V^{-1}$$

where V is the modal matrix of A, that is,

$$\Lambda = V^{-1}AV$$

and $\Lambda = \text{blockdiag}\{\Lambda_1; \Lambda_2\}$ ($\Lambda_1 \in \mathbb{C}^{p\times p}$) is the Jordan canonical form similar to A. Define also a state feedback gain

$$K = \hat{K}C$$

where the gain $\hat{K}$, known in the present work as the aggregated state feedback gain, is chosen in a way that will be discussed later. Also define $\hat{B}$ and $\bar{B}$ as follows:

$$\begin{cases} \hat{B} = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \end{bmatrix} V^{-1}B = CB. \\ \bar{B} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{n-p} \end{bmatrix} V^{-1}B. \end{cases}$$

The closed loop matrix then satisfies

$$\begin{aligned} A + B\hat{K}C &= A + B\hat{K}\begin{bmatrix} \mathbf{I}_p & \mathbf{0} \end{bmatrix} V^{-1} \\ &= V\left(V^{-1}AV + V^{-1}B\hat{K}\begin{bmatrix} \mathbf{I}_p & \mathbf{0} \end{bmatrix}\right)V^{-1} \\ &= V\left(\begin{bmatrix} \Lambda_1 & \mathbf{0} \\ \mathbf{0} & \Lambda_2 \end{bmatrix} + \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-p} \end{bmatrix} V^{-1}B\hat{K}\begin{bmatrix} \mathbf{I}_p & \mathbf{0} \end{bmatrix}\right)V^{-1} \\ &= V\left(\begin{bmatrix} \Lambda_1 & \mathbf{0} \\ \mathbf{0} & \Lambda_2 \end{bmatrix} + \begin{bmatrix} \hat{B}\hat{K} & \mathbf{0} \\ \bar{B}\hat{K} & \mathbf{0} \end{bmatrix}\right)V^{-1} \\ &= V\begin{bmatrix} \Lambda_1 + \hat{B}\hat{K} & \mathbf{0} \\ \bar{B}\hat{K} & \Lambda_2 \end{bmatrix} V^{-1} \end{aligned}$$

which means that $A + B\hat{K}C$ is similar to $\begin{bmatrix} \Lambda_1 + \hat{B}\hat{K} & \mathbf{0} \\ \bar{B}\hat{K} & \Lambda_2 \end{bmatrix}$ and the eigenvalues of Λ_2 are invariant by the feedback $K = \hat{K}C$ and by the way allows to achieve a partial pole placement.

2.2 Partial $\mathcal{D}_R$-stabilization

It is always possible to place p poles at precise arbitrary locations in the complex plane. Many techniques exist (see for instance [15]). However, the present work does not aim at strictly assigning the poles so a greater attention is brought to the non strict placement in the sequel.

Consider the following Hermitian matrix:

$$R = R^* = \begin{bmatrix} R_{00} & R_{10} \\ R_{10}^* & R_{11} \end{bmatrix} \in \mathbb{C}^{2d\times d}, \quad R_{11} \in \mathbb{C}^{d\times d} \geq 0.$$

From this matrix, consider the region $\mathcal{D}_R$ that complies with the following definition

$$\mathcal{D}_R = \left\{ s \in \mathbb{C} \ : \ R_{00} + (R_{10}s)^H + R_{11}s^*s < 0 \right\}. \tag{1}$$

This definition was proposed in [16] except that R is here assumed to be complex (corresponding to regions which are not necessarily symmetric with respect to the real axis). The set of such regions (sometimes called EMI-regions for Ellipsoidal Matrix Inequality) is equivalent to the set of LMI-regions defined in [14]. Hence, they are open convex regions. As examples of practical regions, one can quote discs, half-planes ($d = 1$), sectors, insides of ellipses, strips ($d = 2$), ...

A matrix A is said to be $\mathcal{D}_R$-stable when all its eigenvalues lie inside the region $\mathcal{D}_R$ or in other words, if and only if there exists an Hermitian positive definite matrix P such that matrix A satisfies the condition [16]

$$R_{00} \otimes P + (R_{10} \otimes (PA))^H + R_{11} \otimes (A^* PA) < 0. \tag{2}$$

If the objective is to achieve a partial pole placement in $\mathcal{D}_R$, then the aggregated state feedback gain $\hat{K}$ has to be chosen in such a way that it satisfies

$$R_{00} \otimes \hat{P} + (R_{10} \otimes (\hat{P}(\hat{A} + \hat{B}\hat{K})))^H + R_{11} \otimes ((\hat{A} + \hat{B}\hat{K})^* \hat{P}(\hat{A} + \hat{B}\hat{K})) < 0. \tag{3}$$

for some matrix $\hat{P} = \hat{P}^* > 0$ where the pair $(\hat{A},\ \hat{B})$ defines the aggregated system obtained from the overall system (A, B) with $\hat{A}$ given by

$$\hat{A} \quad = \quad CAC^+ = \Lambda_1$$

(C^+ is some pseudo-inverse of C:

$$C^+ = V \begin{bmatrix} \mathbf{I}_p \\ \mathbf{0} \end{bmatrix} \qquad).$$

Deduced from the above reasoning, the next theorem states the result on partial pole placement:

Theorem 1 *There exists an aggregated state feedback gain that assigns p poles in $\mathcal{D}_R$ if and only if there exist an Hermitian matrix $\hat{X}$ and a matrix $\hat{S}$ such that the LMI*

$$\begin{bmatrix} R_{00} \otimes \hat{X} + (R_{10} \otimes ((\hat{A}\hat{X} + \hat{B}\hat{S})))^H & Z^* \otimes ((\hat{A}\hat{X} + \hat{B}\hat{S})^*) \\ Z \otimes ((\hat{A}\hat{X} + \hat{B}\hat{S})) & -\mathbf{I}_d \otimes \hat{X} \end{bmatrix} < 0 \tag{4}$$

where Z is deduced from the Choleski factorization $R_{11} = Z^*Z$, is satisfied. In this event, a suitable aggregated state feedback gain is

$$\hat{K} = \hat{S}\hat{X}^{-1}. \tag{5}$$

Proof: since $\hat{X}$ is positive definite from (4), it is nonsingular and thus invertible. Define P as $P = X^{-1}$. Apply variable change (5) on (4). Left and right-multiply the obtained inequality by P where P defined as:

$$\mathsf{P} = \begin{bmatrix} \mathbf{I}_d \otimes \hat{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_d \otimes \hat{P} \end{bmatrix}.$$

Finally apply Schur's complement argument to recover (3).
The reasoning can be reversed. □

Note that some solution to (4) (*i.e.* some suitable matrices $\hat{X}$ and $\hat{S}$) can be found with the help of classical software, based upon interior-point methods [17]. Indeed, several LMI toolboxes are now available and can be used to reach such a solution (see [18] for example).

Remark 2 *Note that if R is complex (*i.e. *if $\mathcal{D}_R$ is not symmetric with respect to the real axis), then so are X, $\hat{K}$ and K. Thus, the problem is solved, but in the set of complex matrices. One will come back further on this point in another remark.*

3 Pole placement in a union of regions

Assume now that the region of interest is actually a region, denoted by $\mathcal{D}$, defined as follows:

$$\mathcal{D} = \bigcup_{k=1}^{q} \mathcal{D}_{R_k}$$

where each subregion $\mathcal{D}_{R_k}$ is a region defined as in (1), *i.e.* characterized by a matrix R_k:

$$\mathcal{D}_{R_k} = \left\{ s \in \mathbb{C} \ : \ R_{00_k} + (R_{10_k} s)^H + R_{11_k} s s^* < 0 \right\}.$$

Then the pole placement in region $\mathcal{D}$ can be performed by successive partial pole placement in the subregions $\mathcal{D}_{R_k}$, for $k = 1, \dots q$. This can be achieved according to the following algorithm :

Algorithm

- **Step 1 :** Let $k = 0$ and $A_0 = A$.
- **Step 2:** Let $k = k + 1$
- **Step 3 :** Compute V_{k-1} and Λ_{k-1} such as $V_{k-1}A_{k-1} = \Lambda_{k-1}V_{k-1}$ and rearrange Λ_{k-1} as follows

$$\Lambda_{k-1} = \begin{bmatrix} \Lambda_{k-1}^1 & \mathbf{0} \\ \mathbf{0} & \Lambda_{k-1}^2 \end{bmatrix}$$

where Λ_{k-1}^1 contains the p_k eigenvalues to be shifted into the subregion $\mathcal{D}_{R_k}$. It is worth noting that the matrix V_{k-1} has to be rearranged accordingly to Λ_{k-1}. The matrix C_{k-1} is then constructed as

$$C_{k-1} = \begin{bmatrix} \mathbf{I}_{p_k} & \vdots & \mathbf{0} \end{bmatrix} V_{k-1}^{-1}$$

- **Step 4 :** Compute $\hat{A}_{k-1} = C_{k-1}A_{k-1}C_k^+$ and $\hat{B}_{k-1} = C_{k-1}B$
- **Step 5 :** Find a matrix $\hat{K}_k$ such that there exists $\hat{P}_k > 0$ satisfying

$$R_{00_k} \otimes \hat{P}_k + (R_{10_k} \otimes (\hat{P}_k(\hat{A}_{k-1} + \hat{B}_{k-1}\hat{K}_k)))^H + \\ R_{11_k} \otimes ((\hat{A}_{k-1} + \hat{B}_{k-1}\hat{K}_k)^* \hat{P}_k (\hat{A}_{k-1} + \hat{B}_{k-1}\hat{K}_k)) < 0.$$

i.e. find $\hat{S}_k$ and $\hat{X}_k = \hat{P}_k^{-1}$ satisfying (4) with $\hat{S} = \hat{S}_k$, $\hat{X}_k = \hat{X}_k$, $\hat{A} = \hat{A}_{k-1}$, $\hat{B} = \hat{B}_{k-1}$ and $R_{hl} = R_{hl_k}$ and then deduce $\hat{K}_k = \hat{S}_k\hat{X}_k^{-1}$ according to Theorem 1.

- **Step 6 :** Compute the state feedback gain at step k as

$$K_k = \hat{K}_k C_{k-1}$$

and the closed loop state matrix at step k as

$$A_k = A_{k-1} + BK_k$$

- **Step 7 :** if $k = q$ then go to **Step 8** else go to **Step 2**

- **Step 8 :** Compute the state feedback

$$K = \sum_{k=1}^{p} K_k$$

that achieves the desired pole placement in the specified union of regions.

- **Step 9 :** Stop

Remark 3 *Going on with remark 2, if some R_k is complex, so is matrix K_k. However, in practical problems of control, the global union $\mathcal{D}$ is symmetric. It means that if $\mathcal{D}_k$ is a non symmetric subregion taking part in the union $\mathcal{D}$, so is $\mathcal{D}_h$, the reflection of $\mathcal{D}_k$ with respect to the real axis. As a consequence, K_k and K_h are conjugate and the final matrix K is real.*

4 *A posteriori* robust analysis

This section aims at analyzing the robust $\mathcal{D}$-stability of the obtained closed-loop matrix with respect to a norm-bounded uncertainty. Once the feedback gain K is computed, it is assumed that the uncertain closed-loop matrix reads :

$$\tilde{A}_c = A + BK + JEL = A_c + JEL. \tag{6}$$

To achieve this analysis, we recall a result adapted from [19]. Region $\mathcal{D}$ defined in section 3 may often, in practical cases, comply with the definition of the so-called EEMI-regions (see [19, 20] for more details). Assuming that this generic compatibility between EEMI-formulation and the description of $\mathcal{D}$ holds, some bound on the value of ρ ensuring the robust $\mathcal{D}$-stability of $\tilde{A}_c$ can be computed owing to the next theorem:

Theorem 4 *[19, 20] Let $\tilde{A}_c^{n\times n}$ be a matrix as defined above and $\mathcal{D}$ be a region of the complex plane as defined in Section 3. $\tilde{A}_c$ is $\mathcal{D}$-stable for any E such that $||E||_2 \leq \rho$ if there exists a set of m Hermitian positive definite matrices P_k, $k = 1, ..., m$ such that:*

$$\left[\begin{array}{c|c|c|c} \mathbf{I}_{dn} & \mathbf{I}_d \otimes A' & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_d \otimes J' & \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{array}\right] Q \left[\begin{array}{c|c|c} \mathbf{I}_{dn} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{I}_d \otimes A & \mathbf{I}_d \otimes J & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I} \end{array}\right] < 0 \tag{7}$$

with:

$$Q = \left[\begin{array}{c|c|c} \sum_{k=1}^{q}(R_k \otimes P_k) & \mathbf{0} & \begin{array}{c} \mathbf{I}_d \otimes L' \\ \hline \mathbf{0} \end{array} \\ \hline \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \hline \mathbf{I}_d \otimes L \mid \mathbf{0} & \mathbf{0} & -\dfrac{\mathbf{I}}{\rho^2} \end{array}\right]$$

Note that a solution P_k, $k = 1, \ldots, q$, to (7) can be obtained by using LMI software. Moreover, the LMI machinery enables to reach the maximum value of ρ under convex constraint (7) [18]. If $d = 1$, then the optimal value of ρ is the so-called quadratic $\mathcal{D}$-stability radius. If futhermore $d = q = 1$, it is the so-called complex $\mathcal{D}$-stability radius [21].

5 Illustrative example

This model used in this section is borrowed from the literature and corresponds to a VTOL helicopter. The placement achieved here is not realistic but aims at illustrating the presented work. The considered pair of matrices is:

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0428 & -0.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix} ; \quad B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}$$

The open loop spectrum is $\{-1.882; 0.5915 \pm 0.6776\mathbf{i}; -0.0483\}$. The region $\mathcal{D}$ is chosen as the union of the two discs centered around $(-7;0)$ and $(-20;0)$ and with respective radii 2 and 1. Three poles are expected to lie in the right disc whereas the remaining pole is expected to be shifted into the left disc.

Applying the proposed algorithm , after a first iteration, the feedback gain

$$K_1 = \begin{bmatrix} 0.3377 & 1.1743 & 2.0230 & 2.3983 \\ 0.4577 & 1.3700 & 0.0048 & -5.5256 \end{bmatrix}$$

is computed, which leads to a closed-loop matrix $A_1 = A + BK_1$ with spectrum $\{-0.0483; -5.9481 \pm 1.1132\mathbf{i}; -5.9630\}$ as illustrated on figure 6. Three poles are shifted into the right disc.

The second iteration of the algorithm enables to compute

$$K_2 = \begin{bmatrix} -72.2945 & -9.2494 & -10.9954 & 2.3116 \\ 140.5602 & 17.9834 & 21.3782 & -4.4943 \end{bmatrix}$$

which leads to a global feedback gain:

$$K = K_1 + K_2 = \begin{bmatrix} -71.9567 & -8.0751 & -8.9725 & 4.7098 \\ 141.0179 & 19.3534 & 21.3830 & -10.0199 \end{bmatrix}.$$

The final closed loop matrix is then

$$A_c = A + BK = \begin{bmatrix} -7.0226 & -0.1356 & -0.1833 & -0.1373 \\ -1325.6514 & -175.5677 & -194.1454 & 88.7466 \\ 1030.4718 & 131.8393 & 144.8307 & -69.5675 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix}$$

whose spectrum is $\{-19.9005; -5.9481 \pm 1.1132\mathbf{i}; -5.960\}$, as shown by figure 6.

Assume now that this closed-loop matrix is affected a norm-bounded uncertainty as in equation (6) with:

$$J = \begin{bmatrix} -0.0366 \\ 0.0428 \\ 0.1002 \\ 0.0000 \end{bmatrix} ; \quad L = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

(note that the uncertainty E is actually a scalar in this case). Maximizing ρ under constraint (7) in virtue of theorem 4 leads to the optimal value $\rho^\star = 22.0784$. If the poles of uncertain matrices are plotted for 500 random instances of real matrices E such that $||E||_2 \leq \rho^\star$, one gets some pole migration such as the one given by figure 6 which highlights the relevance of our analysis.

In the same way, it is possible to analyze the fragility induced by the computed feedback gain. For this, just assume that K is subject to an additive uncertainty E such that the closed-loop matrix becomes $A + BK + BE$. It complies with formulation (6) with $J = B$ and $L = \mathbf{I}_3$. In this case, in virtue of theorem 4, one gets $\rho^\star = 0.0285$ and plotting an "uncertain" root-locus as for the previous analysis leads to figure 6 which highlights the tightness of the bound value.

6 Conclusion

In this paper, a technique was proposed to achieve a partial pole placement in a region by static state feedback. From this technique, a method to compute a state feedback gain that assigns a complete pole placement in a union of several possibly disjoint and non symmetric subregions is deduced. With such a method the distribution of the closed-loop spectrum in the various subregions can be arbitrarily chosen. Besides, applicable existing tools were recalled in order to take the robustness of such a placement against a norm-bonded uncertainty into account.

As further extensions, the case where the state and input matrices are subject to parametric uncertainties could perhaps be taken into account in the design procedure. Finally, another question is to know under which conditions this technique could be adapted to the case where a static output feedback control law is to be determined.

References

[1] W. M. Wonham, On Pole assignment in multi-input controllable linear systems, IEEE Transactions on Automatic Control, 12 (6), (1967) 660–665.

[2] F. Kraus, V. Kucera, Robust assignment of polynomial matrix polytopes, in: European Control Conference ECC'97, Vol. 3, Brussels, Belgium, 1997.

[3] M. Saif, Optimal linear regulator pole placement by weight selection, International Journal of Control 50 (1) (1989) 399–414.

[4] N. H. Amin, Optimal pole shifting for continuous multivariable linear systems, Int. Journal of Control 41 (4) (1985) 701–707.

[5] B. C. Moore, On the flexibility offered by state feedback in multivariable systems beyond closed loop eigenvalue assignment, IEEE Transactions on Automatic Control 21 (1976) 689–692.

[6] D. Mehdi, E. Ostertag, Optimal discrete-time pole placement by weight selection, Control - Theory and Advanced Technology 7 (4) (1991) 683–693.

[7] O. A. Solheim, Design of optimal control system with prescribed eigenvalues, International Journal of Control 15 (1) (1972) 143–160.

[8] J. Medanic, H. S. Tharp, W. R. Perkins, Pole placement by performance criterion modification, IEEE Transactions on Automatic Control 33 (5) (1988) 469–472.

[9] D. Arzelier, J. Bernussou, G. Garcia, Pole assignment of linear uncertain systems in a sector via a Lyapunov-type approach, IEEE Transactions on Automatic Control 38 (7) (1993) 1128–1132.

[10] S. O. R. Moheimani, I. R. Petersen, Robust control with pole placement in a sector for a class of uncertain systems, in: European Control Conference ECC'95, 1995.

[11] K. Furuta, S.-B. Kim, Pole assignment in a specified disk, IEEE Transactions on Automatic Control 32 (5) (1987) 423–427.

[12] G. Garcia, J. Bernussou, Pole assignment of uncertain systems in a specified disk by state feedback, IEEE Transactions on Automatic Control 40 (1) (1995) 184–190.

[13] G. Garcia, J. Daafouz, J. Bernussou, Ouput feedback disk pole assignment for systems with positive real uncertainty, IEEE Transactions on Automatic Control 41 (9) (1996) 1385–1391.

[14] M. Chilali, P. Gahinet, H_∞ Design with Pole Placement Constraints : An LMI Approach, IEEE Transactions on Automatic Control 41 (3) (1996) 358–367.

[15] A. N. Andry, E. Y. Shapiro, J. C. Chung, Eigenstructure assignment for linear systems, IEEE Transactions on Aerospace and Electronic Systems 19 (5) (1983) 711–729.

[16] D. Peaucelle, D. Arzelier, O. Bachelier, J. Bernussou, A new robust D-stability condition for real convex polytopic uncertainty, Systems & Control Letters 40 (1) (2000) 21–30.

[17] Y. Nesterov, A. Nemirovskii , Interior-point polynomial methods in convex programming, SIAM Studies in Applied Mathematics 13 (1994).

[18] P. Gahinet, A. Nemirovskii, J. Laub, M. Chilali, LMI Control Toolbox , The MathWorks Inc. (1995).

[19] J. Bosche, O. Bachelier, D. Mehdi, Robustness bounds for matrix D_U-stability, in: European Control Conference ECC'03, Somewhere, 2003.

[20] J. Bosche, O. Bachelier, D. Mehdi, An approach for robust matrix root-clsutering in a union of regions, IMA 38 (7) (1993) 1128–1132.

[21] L. Qiu, B. Bernhardsson, A. Rantzer, E. J. Davison, P. M. Young, J. C. Doyle, A formula for computation of the real stability radius, Automatica 31 (6) (1995) 879–890.

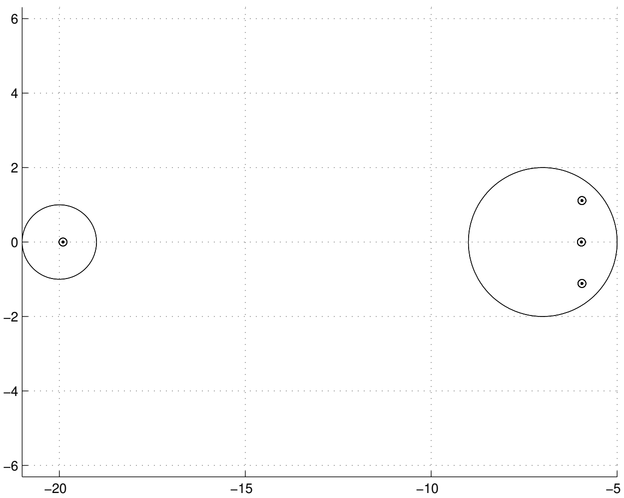

Figure 1: root locus for matrix A_1

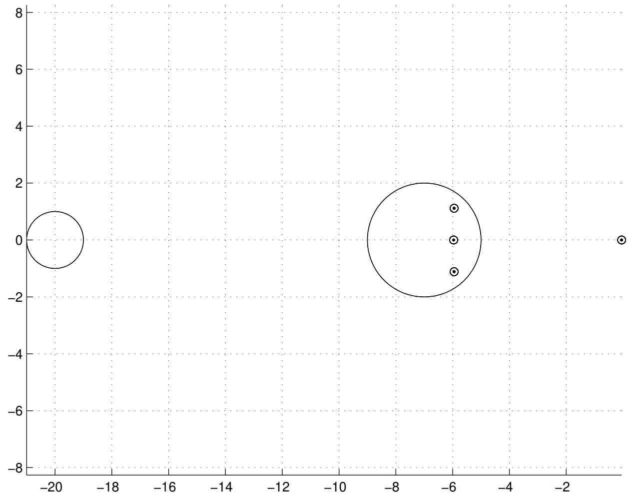

Figure 2: root locus for matrix A

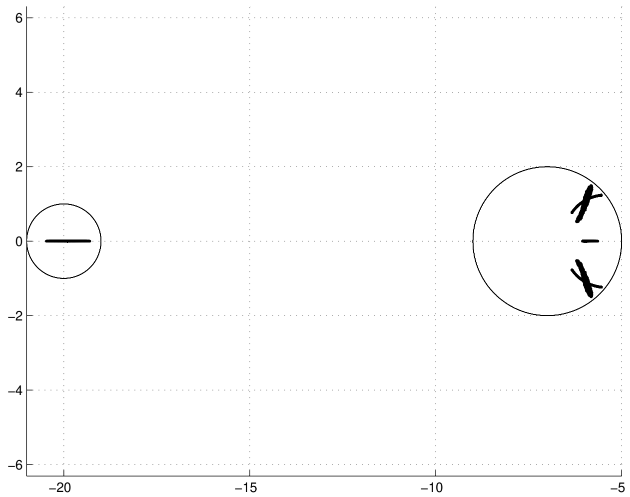

Figure 3: root locus for matrix $\tilde{A}c$ for 500 instances of real E such that $||E||_2 \leq 22.0784$

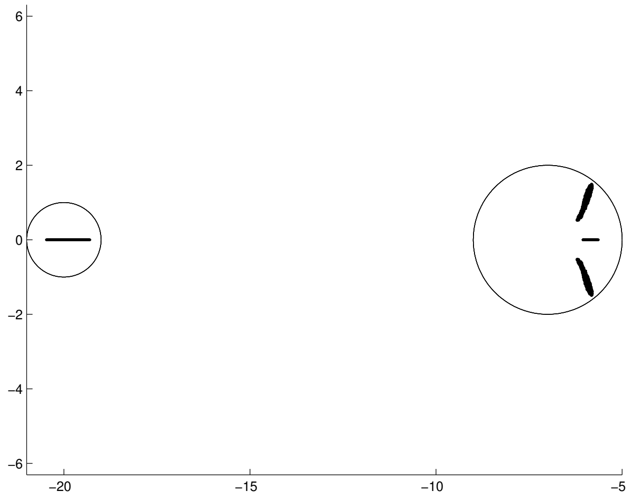

Figure 4: root locus for matrix $\tilde{A}c$ for 500 instances of real E such that $||E||_2 \leq 0.0285$