Robust matrix $\mathcal{D}_U$-stability analysis

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Abstract

In this paper, the problem of robust matrix root-clustering is addressed. The studied matrices are subject to both polytopic and unstructured uncertainty. An original point is the large choice of clustering regions enabled by the proposed approach since these regions can be unions of possibly disjoint and nonsymmetric subregions of the complex plane. The precise purpose is, considering a specified polytope, to determine the greatest robustness bound on the unstructured uncertainty such that robust matrix root-clustering is ensured. To reduce conservatism in the derivation of the bound, the reasonning relies on a framework based upon parameter-dependent Lyapunov functions. The bound value is computed by solving an $\mathcal{LMI}$ *problem.*

1 Introduction

The aim of this paper is to deal with some analytical problems encountered in the study of linear state-space models where some specified performances, obviously including stability, are required and where those performances are expected to be preserved in the presence of model uncertainties. Nominal transient performances might be obtained by assigning closed-loop state matrix eigenvalues that strongly influence settling time, damping ratio and so on. Nevertheless, rather than to require most of the flexibility offered by the multivariable control law by assigning a strict spectrum, it may be more discerning to locate the desired eigenvalues in some appropriate areas of tolerance [1, 2]. Furthermore, a strict placement seems utopian while the model is just an approximation of the real system behaviour. In practice, some uncertainties on the model exist, due, for example, to some unprecisely known physical parameters, neglected dynamics, ... Then a point is raised: how to get both required performances and robustness against uncertainty while computing a pole placement ? To this crucial question, some authors answer by reaching some satisfying nominal performances through a pole placement while improving an index of closed-loop system robustness owing to the degrees of freedom provided by eigenvectors and sets of tolerance [1, 2]. Such works require to define analytical robustness criteria in the sense of eigenvalue location. Finding a suitable robust transient performance criterion

is precisely the point emphasized in the paper.
The uncertainty considered in this paper is both polytopic and unstructured. We aim to derive a bound on the norm of the second uncertainty, such that some matrix root-clustering is ensured with respect to both uncertainties. In this work, the clustering region can be nonconnected since it can be the union of possibly disjoint and nonsymmetric subregions of the complex plane. This is of practical interest in the study of multi-time scale systems (for example, two separate dynamics can be considered: the dominant poles and the fast ones). It can also be used to specify a robust damping ratio by choosing two subregions, symmetric to each other with respect to the real axis and each of them containing complex eigenvalues (see [3] for more details).
A lot of investigations have been led in the area of robust stability or performance analysis against unstructured or structured uncertainty. However, even when only stability is concerned, the proposed conditions are conservative. Actually, proving robust stability over a polytope is known to be a $\mathcal{NP}$-hard problem. That is why only sufficient conditions are proposed and it matters to derive numerically tractable conditions as close as possible to the ideal non-conservative one. Many references deserve to be quoted such as the ones relevant to algebraic approach (see [4] and references therein) or the work related to μ/k_m theory [5, 6]. Unfortunately, the efficiency of these techniques may sometimes be reduced by heavy computations or restrictive assumptions. This is even truer when some performance level is required. A large amount of contributions has been achieved in the area of Lyapunov's theory, using the well known concept of quadratic stability [7, 8, 9] and guaranteed cost [10, 11]. Although the correponding techniques lead to nice conditions (based on Ricatti equations or $\mathcal{LMI}$ ($\mathcal{L}$inear $\mathcal{M}$atrix $\mathcal{I}$nequalities)) for robust analysis and design, a strong drawback appears: using one single Lyapunov function over the whole polytope obviously yields pessimistic results. A way to try to go round that obstacle is to assess robustness via conditions involving parameter-dependent Lyapunov functions [12, 13, 14]. Notice that such functions prove to be very useful for the synthesis of parameter-dependent controllers [15, 16, 17].
When transient performances are required, results about stability are not general enough and we aim to extend some previously published results relevant to robust matrix root-clustering. The problem of robust root-clustering analysis against polytopic uncertainty via parameter-dependent Lyapunov functions has been investigated in [18] and later adapted to nonconnected regions in [19]. We will largely refer to these works in the sequel since in this paper, we propose an $\mathcal{LMI}$ condition to test the robust matrix root-clustering against polytopic uncertainty through a parameter-dependent Lyapunov approach, while the clustering region is a union of convex subregions.
However, as mentionned above, another additive unstructured uncertainty also affects the system model. To test robust matrix root-clustering against an additive uncertainty (first assuming that there is no other polytopic uncertainty), the concept of robustness bound can be used. These are bounds on the uncertainty domain that ensure robust root-clustering. Such a topic has been largely investigated in the last two decades for both unstructured uncertainty and affine structured uncertainty [20]. The first result was proposed in the case where the clustering region is a vertical half plane (stability and relative stability bounds [21]). Those results were later improved in [22, 23, 24]. They were also extended to classes of regions that are usually connected (often even convex) and symmetric with respect to the real axis (let us just say "symmetric") [25, 26, 27, 28, 29, 30]. These bounds are deduced from necessary and sufficient conditions for nominal matrix root-clustering based, for example, on generalised Lyapunov equations [31], on the notion of logarithmic norms [32] or on $\mathcal{LMI}$ framework [33, 34]. Derivation of robustness bounds related to nonconnected regions was a bit handled for unions of disjoint disks and unstructured uncertainty in [35]. It was treated in a more general manner in [36, 3, 37] through a quadratic approach.
To sum up this introduction, it can be said that, on one hand, some results have been proposed to tackle the problem of robust matrix root-clustering against polytopic uncertainty via $\mathcal{LMI}$ conditions involving parameter-dependent Lyapunov functions; on the other hand, the robust pole-clustering against unstructured or structured additive uncertainty has been addressed owing to the concept of

robustness bound. In our work, we associate the two notions. The uncertain matrix is affected by both polytopic and additive unstructured uncertainties. We derive a bound on the additive uncertainty such that the eigenvalues of the uncertain matrix lie in a specified clustering region in the presence of both uncertainties. Moreover, the general case of unions of convex subregions is considered. Furthermore, the bound value is directly obtained by optimizing a linear criterion under $\mathcal{LMI}$ constraints.

The paper is organized as follows : after this introduction, a large section is dedicated to preliminaries and problem statement including some root-clustering concepts, namely the "matrix $\mathcal{D}_R$-stability" and "$\mathcal{D}_U$-stability". In the third section, the problem of robust matrix $\mathcal{D}_R$ (or $\mathcal{D}_U$)-stability against (only) polytopic uncertainty is tackled. In the fourth section that constitutes the most original contribution, we explain how to derive a robust $\mathcal{D}_R$ (or $\mathcal{D}_U$)-stability bound in the presence of both uncertainties. Some numerical illustrations are given in the fifth section before concluding in the sixth one.

Notations : We denote by M^*, the transpose conjugate of M, by M^{-*}, matrix M^{-1^*}, by $\tilde{z}$ the conjugate of the scalar number z, by $Re(z)$ its real part and by $Im(z)$ its imaginary part. The Kronecker product is denoted by $\otimes$. $||M||_2$ is the 2-norm of matrix M induced by the Euclidean vector-norm. $\mathbb{I}_n$ is the identity matrix of order n, $\mathbb{O}_{u,v}$ (resp. $\mathbb{O}_u$) is a null matrix of dimension $u \times v$ (resp. $u \times u$) and $\mathbb{1}_{u,v}$ is a matrix of dimension $u \times v$ whose all entries equal 1. Matrix inequalities are considered in the sense of Löewner *i.e.* "< 0" ("≤ 0") means negative (semi-negative) definite and "> 0" ("≥ 0") positive (semi-positive) definite. HPD stands for Hermitian Positive Definite and PDL for Parameter-Dependent Lyapunov. The set $\mathcal{A}/\mathcal{B}$ is the subset of $\mathcal{A}$ obtained by removing from $\mathcal{A}$ the elements of $\mathcal{B}$. At last $\mathbf{i}$ denotes the imaginary unit.

2 Preliminaries and problem statement

In this section, we give preliminary details about the clustering regions we consider as well as the corresponding matrix root-clustering conditions. The section ends with the formulation of our precise aim including the presentation of the uncertain model.

2.1 Lexical introduction

In this paragraph, we introduce some vocabulary. We deal with two classes of clustering regions: the first one is a set of so-called $\mathcal{EMI}$ ($\mathcal{E}$llipsoidal $\mathcal{M}$atrix $\mathcal{I}$nequality)-regions introduced in [18, 38]. We denote any element of this set by a generic name : $\mathcal{D}_R$. The second class is the set of some nonconnected regions, here called $\mathcal{EEMI}$ ($\mathcal{E}$xtended $\mathcal{E}$llipsoidal $\mathcal{M}$atrix $\mathcal{I}$nequality)-regions and defined as the union of several $\mathcal{EMI}$-regions. Their generic name is $\mathcal{D}_U$. A matrix is said to be $\mathcal{D}_R$-stable when all its eigenvalues lie in an $\mathcal{EMI}$-region $\mathcal{D}_R$. In the same way, a matrix is said to be $\mathcal{D}_U$-stable when its spectrum lies in $\mathcal{D}_U$, an $\mathcal{EEMI}$-region.

2.2 Matrix $\mathcal{D}_R$-stability

We here recall the characterization of an $\mathcal{EMI}$-region $\mathcal{D}_R$ as well as a condition for nominal matrix root-clustering in such a region. Those results are due to [18, 38], following preliminary ideas presented in [13].

2.2.1 Characterization of an $\mathcal{EMI}$-region $\mathcal{D}_R$

Definition 1 *: Let $R \in \mathbb{C}^{2d \times 2d}$ be the Hermitian matrix defined by:*

$$\left\{\begin{array}{l} R = R^* = \left[\begin{array}{cc} R_{00} & R_{10} \\ R_{10}^* & R_{11} \end{array}\right] \\ \\ R_{11} \geq 0 \in \mathbb{C}^{d\times d} \end{array}\right. \quad (1)$$

The set of points $\mathcal{D}_R$ defined by:

$$\mathcal{D}_R = \{z \in \mathbb{C} \mid f_{\mathcal{D}_R}(z) = \left[\begin{array}{cc} \mathbb{I}_d & \tilde{z}\mathbb{I}_d \end{array}\right] R \left[\begin{array}{c} \mathbb{I}_d \\ z\mathbb{I}_d \end{array}\right] < 0\} \quad (2)$$

is called an $\mathcal{EMI}$-region of degree d.

Such a region is convex. For a real R, this formulation is close to the one of $\mathcal{LMI}$ regions (see [18] for more details). Here, we allow R to be complex, *i.e.* $\mathcal{D}_R$ to be nonsymmetric. The set of $\mathcal{EMI}$-regions includes, for instance, shifted and nonsymmetric half planes, classical and hyperbolic sectors, vertical and horizontal strips, disks or ellipses the centers of which are not necessarily on the real axis and so on...

Remark 2 *: Once developed, the characteristic function $f_{\mathcal{D}_R}$ can be written:*

$$f_{\mathcal{D}_R}(z) = R_{00} + R_{10}z + R_{10}^*\tilde{z} + R_{11}z\tilde{z} \quad (3)$$

This is the expression which is used in [18] and we will use either this expression or the one given in (2).

2.2.2 Condition for matrix $\mathcal{D}_R$-stability

In this paragraph, we still refer to [18, 38] and recall a necessary and sufficient condition for the eigenvalues of a complex matrix to lie in an $\mathcal{EMI}$-region.

Theorem 3 *: Let $\mathcal{D}_R$ be an $\mathcal{EMI}$-region defined as in definition 1. A matrix $A \in \mathbb{C}^{n\times n}$ is $\mathcal{D}_R$-stable if and only if there exists an HPD matrix $P \in \mathbb{C}^{n\times n}$ such that the following $\mathcal{LMI}$ holds:*

$$\mathbb{R}(A, P) = \left[\begin{array}{cc} \mathbb{I}_{dn} & \mathbb{I}_d \otimes A^* \end{array}\right] (R \otimes P) \left[\begin{array}{c} \mathbb{I}_{dn} \\ \mathbb{I}_d \otimes A \end{array}\right] < 0 \quad (4)$$

Proof of theorem 3: see [18, 38]. For a real R, this result is analogous to [33, theorem 2.2] that is relevant to $\mathcal{LMI}$-regions.

Remark 4 *: Taking remark 2 into account, inequality (4) can be written:*

$$\mathbb{R}(A, P) = R_{00} \otimes P + R_{10} \otimes (PA) + R_{10}^* \otimes (A^*P) + R_{11} \otimes (A^*PA) < 0 \quad (5)$$

which can be easily deduced from the expression of $f_{\mathcal{D}_R}(z)$ given in (3) by the substitution:

$$(1\,;\, z\,;\, \tilde{z}\,;\, z\tilde{z}) \leftrightarrow (\otimes P\,;\, \otimes PA\,;\, \otimes A^*P\,;\, \otimes(A^*PA)) \quad (6)$$

2.3 Matrix $\mathcal{D}_U$-stability

In this part, we extend the previous results to the case where the clustering region is an $\mathcal{EEMI}$-region.

2.3.1 Characterization of an $\mathcal{EEMI}$-region $\mathcal{D}_U$

Definition 5 *: Let $\mathcal{R}$ be a set of m Hermitian matrices R_k defined by:*

$$\left\{ \begin{array}{l} R_k = R_k^* = \begin{bmatrix} R_{k_{00}} & R_{k10} \\ R_{k_{10}}^* & R_{k11} \end{bmatrix} \in \mathbb{C}^{2d \times 2d} \\ \\ R_{k_{11}} \geq 0 \in \mathbb{C}^{d \times d} \end{array} \right. \qquad \forall k \in \{1, ..., m\} \tag{7}$$

The set of points $\mathcal{D}_U$ defined by:

$$\mathcal{D}_U = \bigcup_{k=1}^{m} \mathcal{D}_{R_k} \tag{8}$$

where each subregion $\mathcal{D}_{R_k}$ is an $\mathcal{EMI}$-region of degree d_k described by:

$$\mathcal{D}_{R_k} = \{z \in \mathbb{C} \mid f_{\mathcal{D}_{R_k}}(z) = \begin{bmatrix} \mathbb{I}_{d_k} & \tilde{z}\mathbb{I}_{d_k} \end{bmatrix} R_k \begin{bmatrix} \mathbb{I}_{d_k} \\ z\mathbb{I}_{d_k} \end{bmatrix} < 0\} \quad \forall k \in \{1, ..., m\} \tag{9}$$

is called an $\mathcal{EEMI}$-region.

Remark 6 *: In the sequel, it is assumed that $d_k = d$, $\forall k \in \{1, ..., m\}$. It is not restrictive. Indeed, if:*

$$k^\star = \arg \ (\max_{k=1,\ldots,m} d_k) \quad ; \quad d = d_{k^\star} = \max_{k=1,\ldots,m} d_k \quad \text{and} \quad \nu_k = d - d_k \quad \forall k \in \{1, ..., m\}, \tag{10}$$

then the set $\mathcal{R}$ can be artificially modified as follows:

$$R_{k_{00}} := \begin{bmatrix} R_{k_{00}} & \mathbb{O}_{d_k,\nu_k} \\ \mathbb{O}_{\nu_k,d_k} & -\mathbb{I}_{\nu_k} \end{bmatrix} ; \ R_{k_{10}} := \begin{bmatrix} R_{k_{10}} & \mathbb{O}_{d_k,\nu_k} \\ \mathbb{O}_{\nu_k,d_k} & \mathbb{O}_{\nu_k} \end{bmatrix} ; \ R_{k_{11}} := \begin{bmatrix} R_{k_{11}} & \mathbb{O}_{d_k,\nu_k} \\ \mathbb{O}_{\nu_k,d_k} & \mathbb{O}_{\nu_k} \end{bmatrix} \ \forall k \in \{\{1, ..., m\}/k^\star\} \tag{11}$$

2.3.2 Point clustering condition

Before dealing with the problem of matrix eigenvalue-clustering, it matters to express a condition for a point of a complex plane to lie in an $\mathcal{EEMI}$-region $\mathcal{D}_U$.

Theorem 7 *: Let $\mathcal{D}_U$ be an $\mathcal{EEMI}$-region as defined in definition 5. A point of affix $z \in \mathbb{C}$ lies in an $\mathcal{EEMI}$-region $\mathcal{D}_U$ if and only if there exists a $\mathbb{R}^m$-vector $\underline{w}$ with strictly positive entries w_k, $k = 1, ..., m$, satisfying :*

$$f_{\mathcal{D}_U}(\underline{w}, z) = \sum_{k=1}^{m} (w_k f_{\mathcal{D}_{R_k}}(z)) < 0 \tag{12}$$

proof of theorem 7: such a reasonning can be found in [34] but we repeat it for clarity. Indeed, if $z \in \mathcal{D}_{R_l}$, $l \in \{1, ..., m\}$, one can always find a sufficiently large value of w_l as well as sufficiently small values of w_k, $\forall k \in \{\{1, ..., m\}/l\}$ such that (12) holds. Reciprocally, if (12) holds for some suitable vector $\underline{w}$, then at least one of the terms in the sum is negative, which means that z is located inside at least one of the subregions. $\square$

Taking this point clustering condition into account, an $\mathcal{EEMI}$-region can be directly characterized by:

$$\mathcal{D}_U \;=\; \{z \in \mathbb{C} \mid \exists \underline{w} \;=\; \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \in \{\mathbb{R}^+/0\}^m \;:\; f_{\mathcal{D}_U}(\underline{w}, z) \;<\; 0\} \tag{13}$$

The characteristic function $f_{\mathcal{D}_U}(\underline{w}, z)$, defined in (12) and appearing in (13), can be written:

$$f_{\mathcal{D}_U}(\underline{w}, z) \;=\; \begin{bmatrix} \mathbb{I}_d & \tilde{z}(\mathbb{1}_{1,m} \otimes \mathbb{I}_d) \end{bmatrix} U(\underline{w}) \begin{bmatrix} \mathbb{I}_d \\ z(\mathbb{1}_{m,1} \otimes \mathbb{I}_d) \end{bmatrix} \tag{14}$$

with:

$$U(\underline{w}) \;=\; \begin{bmatrix} \displaystyle\sum_{k=1}^{m}(w_k R_{k_{00}}) & [w_k R_{k_{10}}]_{1 \le k \le m} \\ ([w_k R_{k_{10}}]_{1 \le k \le m})^* & \displaystyle\operatorname*{diag}_{k=1,\dots,m} \{w_k R_{k_{11}}\} \end{bmatrix} \tag{15}$$

where:

$$[w_k R_{k_{10}}]_{1 \le k \le m} \;=\; \begin{bmatrix} w_1 R_{1_{10}} & \dots & w_k R_{k_{10}} & \dots & w_m R_{m_{10}} \end{bmatrix} \tag{16}$$

(This kind of notation is largely used in the sequel).

Remark 8 *: Going on with remarks 2 and 4, it can be noticed that:*

$$f_{\mathcal{D}_U}(\underline{w}, z) \;=\; \sum_{k=1}^{m}(w_k(R_{k_{00}} \;+\; R_{k_{10}} z \;+\; R_{k_{10}}^* \tilde{z} \;+\; R_{k_{11}} z\tilde{z})) \;<\; 0 \tag{17}$$

2.3.3 Condition for matrix $\mathcal{D}_U$-stability

We now reformulate a condition proposed in [19] for a matrix to be $\mathcal{D}_U$-stable.

Theorem 9 *: Let $\mathcal{D}_U$ be an $\mathcal{EEMI}$-region as introduced in definition 5. A matrix $A \in \mathbb{C}^{n \times n}$ is $\mathcal{D}_U$-stable if and only if there exist a set $\mathcal{P}$ of m HPD matrices $P_k \in \mathbb{C}^{n \times n}$, $k = 1, \dots, m$, such that:*

$$\mathbb{U}(A, \mathcal{P}) \;=\; \begin{bmatrix} \mathbb{I}_{dn} & \mathbb{1}_{1,m} \otimes \mathbb{I}_d \otimes A^* \end{bmatrix} \mathfrak{U}(\mathcal{P}) \begin{bmatrix} \mathbb{I}_{dn} \\ \mathbb{1}_{m,1} \otimes \mathbb{I}_d \otimes A \end{bmatrix} \;<\; 0 \tag{18}$$

with:

$$\mathfrak{U}(\mathcal{P}) \;=\; \begin{bmatrix} \displaystyle\sum_{k=1}^{m}(R_{k_{00}} \otimes P_k) & [R_{k_{10}} \otimes P_k]_{1 \le k \le m} \\ ([R_{k_{10}} \otimes P_k]_{1 \le k \le m})^* & \displaystyle\operatorname*{diag}_{k=1,\dots,m} \{R_{k_{11}} \otimes P_k\} \end{bmatrix} \tag{19}$$

Remark 10 *: Still going on with remarks 2, 4 and 8, notice that (18) is equivalent to:*

$$\mathbb{U}(A, \mathcal{P}) \;=\; \sum_{k=1}^{m}(R_{k_{00}} \otimes P_k \;+\; R_{k_{10}} \otimes (P_k A) \;+\; R_{k_{10}}^* \otimes (A^* P_k) \;+\; R_{k_{11}} \otimes (A^* P_k A)) \;<\; 0 \tag{20}$$

proof of theorem 9: given in appendix.

2.4 Our purpose

In this part, after a presentation of the uncertainty structure, we state the complete problem we aim to solve.

2.4.1 The uncertain model

Assume that we are interested in the transient performances of a continuous or discrete-time model whose state matrix is:

$$\mathbf{A} = (A + E) \in \mathbb{C}^{n\times n} \tag{21}$$

Matrix E is an unknown additive term taking, for example, neglected dynamics or approximations induced by linearization into account. It is such that $||E||_2 = \bar{\sigma}(E) \leq \rho$. Matrix A is also an uncertain matrix lying somewhere in the polytope $\mathcal{A}$ defined by:

$$\mathcal{A} = \{A = A(\alpha) \,|\, \sum_{j=1}^{N} \alpha_{[j]} A_{[j]} \,;\, \alpha \in \boldsymbol{\alpha} \} \tag{22}$$

where $\boldsymbol{\alpha}$, the set of all possible barycentric coordinates, is itself defined by:

$$\boldsymbol{\alpha} = \{\alpha = \begin{bmatrix} \alpha_{[1]} \\ \vdots \\ \alpha_{[N]} \end{bmatrix} \in \{\mathbb{R}^+\}^N \,|\, \sum_{j=1}^{N} \alpha_{[j]} = 1\} \tag{23}$$

and where the matrices $A_{[j]} \in \mathbb{C}^{n\times n}$, $j = 1, ...N$, are the known vertices of $\mathcal{A}$. This polytopic structure encompasses the classical affine parametric uncertainty [20]:

$$A = A_0 + \sum_{l=1}^{h} (\delta_l E_l) \tag{24}$$

where A_0 is the nominal matrix, where δ_l are unknown bounded real scalar parameters defining an hyperrectangular set and where the matrices E_l specify which entries of A are affected by the parameter variation. For this particular case, $\mathcal{A}$ is an orthotope with $N = 2^h$ vertices.
We conclude this paragraph by noticing that for the practical problems, matrix $\mathbf{A}$ is real although our reasonning is available for complex matrices.

2.4.2 Formulation of the precise problem

In this paper, we consider that the polytope $\mathcal{A}$ is fixed. Let $\mathcal{E}_\rho$ be the set of complex $(n\times n)$-matrices E such that $||E||_2 < \rho$. Since $\mathcal{A}$ is fixed, we aim to find the largest $\mathcal{E}_\rho$ such that $\mathbf{A}$ has all its eigenvalues inside a specified $\mathcal{EEMI}$-region $\mathcal{D}_U$ of the complex plane. In other words, our goal is to compute, through an $\mathcal{LMI}$ approach, $\rho^\star$, the greatest value of ρ such that the $\mathcal{D}_U$-stability of $\mathbf{A}$ is ensured, whatever the value of $\alpha \in \boldsymbol{\alpha}$ is. $\rho^\star$ is called a robustness bound or a robust $\mathcal{D}_U$-stability bound.
In the next section, we will first consider $\mathcal{E}_0$, *i.e.* we will only take the polytopic uncertainty into account. We will recall the result of [18] about matrix $\mathcal{D}_R$-stability showing the interest of using a parameter-dependent Lyapunov approach and extend it to $\mathcal{D}_U$-stability. Then, in another section, we will consider both uncertainties together and propose the main contribution of the paper, that is the derivation of the robust $\mathcal{D}_U$-stability bound $\rho^\star$.

Remark 11 *A robust $\mathcal{D}_U$-stability bound is a lower bound of the ideal $\mathcal{D}_U$-stability radius (see [39] for details on the $\mathcal{D}$-stability radius). When only stability is concerned, it is the inverse of an upper bound of μ, the structured singular value [6].*

3 Robust matrix root-clustering: the polytopic case

As explained above, we first omit the additive unstructured uncertainty in this section and just consider the polytopic one. We recall some result proposed in [18, 38] while $\mathcal{D}_R$-stability is concerned and we extend it to $\mathcal{D}_U$-stability.

3.1 Robust $\mathcal{D}_R$-stability of $A(\alpha)$

Polytope $\mathcal{A}$ defined in (22) is said to be robustly $\mathcal{D}_R$-stable if and only if $A(\alpha)$ is $\mathcal{D}_R$-stable whatever the value of $\alpha \in \boldsymbol{\alpha}$ is. In the Lyapunov context, testing this property is equivalent to compute a Lyapunov HPD matrix $P(\alpha)$ (then depending on the uncertainty structure) such that:

$$\mathbb{R}(A(\alpha), P(\alpha)) \; < \; 0 \quad \forall \alpha \in \boldsymbol{\alpha} \tag{25}$$

Unfortunately, condition (25) is very difficult (even impossible) to check over the whole polytope and it matters to find sufficient conditions that are numerically tractable. One way to tackle this problem is to extend the concept of quadratic stability [7] by computing one single Lyapunov matrix P over the whole polytope. $\mathcal{A}$ is said quadratically $\mathcal{D}_R$-stable if and only if there exists an HPD matrix P such that:

$$\mathbb{R}(A(\alpha), P) \; < \; 0 \quad \forall \alpha \in \boldsymbol{\alpha} \tag{26}$$

Quadratic $\mathcal{D}_R$-stability is only sufficient for robust $\mathcal{D}_R$-stability. A tractable necessary and sufficient condition for the polytope to be quadratically $\mathcal{D}_R$-stable is proposed in [18]. Although such a condition is very tractable from a computational point of view, its main drawback is the use of a single matrix P over $\mathcal{A}$. It is why Peaucelle and co-authors [18] also proposed a less drastic condition for testing $\mathcal{D}_R$-stability via the computation of PDL matrices that is recalled here. To properly understand the sequel, we have to recall another condition for nominal $\mathcal{D}_R$-stability.

Theorem 12 *: Let $\mathcal{D}_R$ be an $\mathcal{EMI}$-region. A matrix $A \in \mathbb{C}^{n\times n}$ is $\mathcal{D}_R$-stable if and only if there exist an HPD matrix P as well as a matrix $G_R \in \mathbb{C}^{2dn\times dn}$ such that:*

$$\overline{\mathbb{R}}(A, P, G_R) \; = \; R \otimes P + \begin{bmatrix} \mathbb{I}_d \otimes A^* \\ -\mathbb{I}_d \end{bmatrix} G_R^* + G_R \begin{bmatrix} \mathbb{I}_d \otimes A & -\mathbb{I}_d \end{bmatrix} < 0 \tag{27}$$

Proof of theorem 12: this is some particular application of matrix elimination lemma. An alternative proof can be found in [18].

From a nominal point of view, this condition is of no interest but the flexibility offered by matrix G_R reveals as very useful in a robust context as highlighted in the next theorem [18, 38].

Theorem 13 *: Let $\mathcal{D}_R$ be an $\mathcal{EMI}$-region. The polytope $\mathcal{A}$ defined in (22) is robustly $\mathcal{D}_R$-stable if there exists N HPD matrices $P_{[j]} \in \mathbb{C}^{n\times n}$, $j = 1, ..., N$, as well as a matrix $G_R \in \mathbb{C}^{2dn\times dn}$ such that:*

$$\overline{\mathbb{R}}(A_{[j]}, P_{[j]}, G_R) \; < \; 0 \quad \forall j \in \{1, ..., N\} \tag{28}$$

Proof of theorem 13: see [18].

Condition (28) implies that each matrix $A(\alpha) \in \mathcal{A}$ is proved $\mathcal{D}_R$-stable owing to a Lyapunov matrix $P(\alpha)$ which is a convex combination of the various Lyapunov matrices $P_{[j]}$ computed on the vertices, *i.e.*:

$$P(\alpha) = \sum_{j=1}^{N} (\alpha_{[j]} P_{[j]}) \quad \forall \alpha \in \boldsymbol{\alpha} \tag{29}$$

Hence, the Lyapunov function is parameter-dependent and it can be shown, following arguments of [13], that (28) is necessary for quadratic $\mathcal{D}_R$-stability to be satisfied. It is detailed further in the case of $\mathcal{D}_U$-stability.

3.2 Robust $\mathcal{D}_U$-stability of $A(\alpha)$

This part refers to [19] and is an extension of the results presented in the previous part. We still consider polytope $\mathcal{A}$ defined by (22). It is said robustly $\mathcal{D}_U$-stable when, $\forall \alpha \in \boldsymbol{\alpha}$ is, the spectrum of $A(\alpha)$ is located inside $\mathcal{D}_U$, a specified $\mathcal{EEMI}$-region. In the Lyapunov context, it is equivalent to the existence of a set $\mathcal{P}(\alpha)$ of m HPD PDL matrices $P_k(\alpha)$, $k = 1, ..., m$, such that:

$$\mathbb{U}(A(\alpha), \mathcal{P}(\alpha)) < 0 \quad \forall \alpha \in \boldsymbol{\alpha} \tag{30}$$

Of course, this condition is not tractable from a computational point of view. As for $\mathcal{D}_R$-stability, quadratic $\mathcal{D}_U$-stability consists in considering a set $\mathcal{P}$ of m HPD constant matrices P_k, $k = 1, ..., m$ such that:

$$\mathbb{U}(A(\alpha), \mathcal{P}) < 0 \quad \forall \alpha \in \boldsymbol{\alpha} \tag{31}$$

Some equivalent tractable condition could be an extension of [18], what would be sufficient for robust $\mathcal{D}_U$-stability. Actually, as in the case of $\mathcal{D}_R$-stability, the drawback of this tractable condition is the computation of a single set $\mathcal{P}$ attesting $\mathcal{D}_U$-stability over the whole polytope. To overcome this restriction, we rather propose a PDL counterpart. It is first necessary to extend theorem 12.

Theorem 14 *: Let $\mathcal{D}_U$ be an $\mathcal{EEMI}$-region. A matrix $A \in \mathbb{C}^{n\times n}$ is $\mathcal{D}_U$-stable if and only if there exists a set $\mathcal{P}$ of m HPD matrices P_k, $k = 1, ..., m$ as well as a matrix $G_U \in \mathbb{C}^{(m+1)dn\times mdn}$ such that:*

$$\overline{\mathbb{U}}(A, \mathcal{P}, G_U) = \mathcal{U}(\mathcal{P}) + \begin{bmatrix} \mathbb{1}_{1,m} \otimes \mathbb{I}_d \otimes A^* \\ -\mathbb{I}_{mdn} \end{bmatrix} G_U^* + G_U \begin{bmatrix} \mathbb{1}_{m,1} \otimes \mathbb{I}_d \otimes A & -\mathbb{I}_{mdn} \end{bmatrix} < 0 \tag{32}$$

Proof of theorem 14: given in appendix.

Here again, the interest of G_U appears in a robust context as shown in the next theorem:

Theorem 15 *: Let $\mathcal{D}_U$ be an $\mathcal{EEMI}$-region. The polytope $\mathcal{A}$ defined in (22) is robustly $\mathcal{D}_U$-stable if there exist N sets $\mathcal{P}_{[j]}$, each of them made up by m HPD matrices $P_{[j]_k}$, $j = 1, ..., N$, $k = 1, ..., m$, as well as a matrix $G_U \in \mathbb{C}^{(m+1)dn\times mdn}$ such that:*

$$\overline{\mathbb{U}}(A_{[j]}, \mathcal{P}_{[j]}, G_U) < 0 \quad \forall j \in \{1, ..., N\} \tag{33}$$

Proof of theorem 15: from (33), it comes:

$$\sum_{j=1}^{N} (\alpha_j \overline{\mathbb{U}}(A_{[j]}, \mathcal{P}_{[j]}, G_U)) = \overline{\mathbb{U}}(A(\alpha), \mathcal{P}(\alpha), G_U) < 0 \quad \forall \alpha \in \boldsymbol{\alpha} \tag{34}$$

where $\mathcal{P}(\alpha)$ is the set of m (obviously HPD) matrices $P_k(\alpha)$ defined by:

$$P_k(\alpha) = \sum_{j=1}^{N}(\alpha_{[j]}P_{[j]_k}) \quad \forall\{k;\alpha\} \in \{\{1,...,m\} \times \boldsymbol{\alpha}\ \} \tag{35}$$

Pre and postmultiplying (34) by respectively $\mathbb{S}^*(A(\alpha))$ and $\mathbb{S}(A(\alpha))$ (function $\mathbb{S}(.)$ is defined in (75)) leads to:

$$\mathbb{U}(A(\alpha), \mathcal{P}(\alpha)) < 0 \quad \square \tag{36}$$

Compared to the result of [19], the previous theorem is a bit more general since, the condition proposed in [19] is analogous to the one of theorem 15 but with the following constraint on the structure of G_U:

$$G_U = \begin{bmatrix} [G_{R_k}^{[1]}]_{1\le k\le m} \\ \\ \operatorname*{diag}_{k=1,...,m} \quad \{G_{R_k}^{[2]}\} \end{bmatrix} \tag{37}$$

where $\{G_{R_k}^{[1]}; G_{R_k}^{[2]}\} \in \{\mathbb{C}^{dn\times dn}\}^2\ \forall k$. Notice that the stucture of G_U given in (86) and chosen to achieve the necessity part of the proof of theorem 14 in appendix is also a particular case of structure (37). For this reason, we conjecture: that G_U can be constrained as in (37) with no strong additional conservatism.

It can be noticed from (35) that the Lyapunov matrix relevant to each subregion $\mathcal{D}_{R_k}$ is parameter-dependent so it is a strong reduction of conservatism compared with quadratic $\mathcal{D}_U$-stability. To theoretically guarantee this improvement, we propose the following theorem:

Theorem 16 *: Quadratic $\mathcal{D}_U$-stability is sufficient for PDL $\mathcal{D}_U$-stability to be satisfied.*

Proof of theorem 16: if quadratic $\mathcal{D}_U$-stability is checked, there exists a set $\mathcal{P}$ such that (31) holds. As the uncertainty domain is a bounded set, one can find a scalar number $\varepsilon > 0$ small enough such that:

$$\mathbb{S}^*(A(\alpha))\mathfrak{U}(\mathcal{P})\mathbb{S}(A(\alpha)) + \varepsilon \sum_{k=1}^{m}(\mathbb{Z}^*(A(\alpha))(\operatorname*{diag}_{k=1,...,m} \quad \{P_k\}\ )\mathbb{Z}(A(\alpha))) < 0 \quad \forall \alpha \in \boldsymbol{\alpha} \tag{38}$$

where functions $\mathbb{S}(.)$ and $\mathbb{Z}(.)$ are defined in (75) and (76). At this stage, the reasonning used in the proof of theorem 14 (necessity part) can be slightly modified with substituting A by $A(\alpha)$. Hence, considering this substitution, we have $\overline{\mathbb{U}}(A(\alpha, \mathcal{P}, G_U) < 0$ for any matrix $A(\alpha)$ taken in the polytope and for the choice (86). It also holds for each vertex ($A \leftarrow A_{[j]}$). Then choosing $P_{[j]_k} = P_k \quad \forall\{j;k\} \in \{\{1,...,N\} \times \{1,...,m\}\}$ proves that (33) holds for some sets $\mathcal{P}_{[j]}$ and for some G_U. $\square$

4 Robust matrix root-clustering against both uncertainties

We now consider the whole uncertain matrix $\mathbf{A}$ described in (21). We propose a method to compute a bound $\rho^\star$ so that the $\mathcal{D}_U$-stability of $\mathbf{A}$ is ensured whatever the value of $\alpha \in \boldsymbol{\alpha}$ is and for any matrix $E \in \mathcal{E}_{\rho^\star}$. We first consider the case of $\mathcal{D}_R$-stability. Then, we enlarge our contribution to the more general case of $\mathcal{D}_U$-stability. It is the most original point of our contribution.

4.1 Robust $\mathcal{D}_R$-stability bound

We propose the following theorem:

Theorem 17 : *Let $\mathcal{D}_R$ be an $\mathcal{EMI}$-region. Uncertain matrix $\mathbf{A}$ defined in (21) is $\mathcal{D}_R$-stable if:*

$$\bar{\sigma}(E) \leq \rho^\star = \sqrt{\gamma^\star} \tag{39}$$

with $\gamma^\star$ being the scalar optimal value obtained by solving the following problem :

$$\max_{P_{[1]},\ldots,P_{[N]},G_R,\gamma} \quad \gamma \tag{40}$$

under the $\mathcal{LMI}$ constraints, $\forall j \in \{1,\ldots,N\}$:

$$\left\{\begin{array}{l} \mathbb{M}_R(A_{[j]},P_{[j]},G_R) = \begin{bmatrix} 2\gamma \mathbb{I}_{dn} & \mathbb{O}_{dn} \\ \mathbb{O}_{dn} & \mathbb{O}_{dn} \end{bmatrix} + \overline{\mathbb{R}}(A_{[j]},P_{[j]},G_R) < 0 \\ \mathbb{N}_R(A_{[j]},G_R) = \begin{bmatrix} -\gamma \mathbb{I}_{dn} & \mathbb{O}_{dn} \\ \mathbb{O}_{dn} & -\mathbb{I}_{dn} \end{bmatrix} + \begin{bmatrix} \mathbb{O}_{dn} \\ \mathbb{I}_{dn} \end{bmatrix} G_R^* \begin{bmatrix} \mathbb{I}_{dn} & \mathbb{O}_{dn} \\ \mathbb{I}_d \otimes A_{[j]} & \mathbb{I}_{dn} \end{bmatrix} + \\ \qquad \begin{bmatrix} \mathbb{I}_{dn} & \mathbb{I}_d \otimes A_{[j]}^* \\ \mathbb{O}_{dn} & \mathbb{I}_{dn} \end{bmatrix} G_R \begin{bmatrix} \mathbb{O}_{dn} & \mathbb{I}_{dn} \end{bmatrix} < 0 \end{array}\right. \tag{41}$$

and where $G_R \in \mathbb{C}^{2dn\times dn}$ and the set $\mathcal{P}$ of m HPD matrices $P_k \in \mathbb{C}^{n\times n}$, $k = 1,\ldots,m$, together with $\gamma \in \mathbb{R}$ are the $\mathcal{LMI}$ variables.

proof of theorem 17: the proof is detailed for the more general case of $\mathcal{D}_U$-stability.

4.2 Robust $\mathcal{D}_U$-stability bound

In this part, theorem 17 is extented to matrix root-clustering in an $\mathcal{EEMI}$-region.

Theorem 18 : *Let $\mathcal{D}_U$ be an $\mathcal{EEMI}$-region. Uncertain matrix $\mathbf{A}$ defined in (21) is $\mathcal{D}_U$-stable if:*

$$\bar{\sigma}(E) \leq \rho^\star = \sqrt{\gamma^\star} \tag{42}$$

with $\gamma^\star$ being the scalar optimal value obtained by solving the following problem :

$$\max_{\mathcal{P}_{[1]},\ldots,\mathcal{P}_{[N]},G_U,\gamma} \quad \gamma \tag{43}$$

under the $\mathcal{LMI}$ constraints, $\forall j \in \{1,\ldots,N\}$:

$$\left\{\begin{array}{l} \mathbb{M}_U(A_{[j]},\mathcal{P}_{[j]},G_U) = \begin{bmatrix} 2\gamma m \mathbb{I}_{dn} & \mathbb{O}_{dn,mdn} \\ \mathbb{O}_{mdn,dn} & \mathbb{O}_{mdn} \end{bmatrix} + \overline{\mathbb{U}}(A_{[j]},\mathcal{P}_{[j]},G_U) < 0 \\ \mathbb{N}_U(A_{[j]},G_U) = \begin{bmatrix} -\gamma m \mathbb{I}_{dn} & \mathbb{O}_{dn,mdn} \\ \mathbb{O}_{mdn,dn} & -\mathbb{I}_{dmn} \end{bmatrix} + \begin{bmatrix} \mathbb{O}_{dn,mdn} \\ \mathbb{I}_{mdn} \end{bmatrix} G_U^* \begin{bmatrix} \mathbb{I}_{dn} & \mathbb{O}_{dn,mdn} \\ \mathbb{Z}(A_{[j]}) & \mathbb{I}_{mdn} \end{bmatrix} + \\ \qquad \begin{bmatrix} \mathbb{I}_{dn} & \mathbb{Z}^*(A_{[j]}) \\ \mathbb{O}_{mdn,dn} & \mathbb{I}_{mdn} \end{bmatrix} G_U \begin{bmatrix} \mathbb{O}_{mdn,dn} & \mathbb{I}_{mdn} \end{bmatrix} < 0 \end{array}\right. \tag{44}$$

where $G_U \in \mathbb{C}^{(m+1)dn \times mdn}$ and the N sets $\mathcal{P}_{[j]}$, $j = 1, ..., N$, each of them made up by m HPD matrices $P_{[j]_k} \in \mathbb{C}^{n\times n}$, $k = 1, ..., m$, together with $\gamma \in \mathbb{R}$ are the $\mathcal{LMI}$ variables, and where the matrix function $\mathbb{Z}(\cdot)$ is defined in (76).

Proof of theorem 18: first calculate the following convex combinations:

$$\left\{\begin{array}{l} \mathbb{M}_U(A(\alpha), \mathcal{P}(\alpha), G_U) \;=\; \sum_{j=1}^{N} (\alpha_{[j]} \mathbb{M}_{U_{[j]}}) \;<\; 0 \\ \\ \mathbb{N}_U(A(\alpha), G_U) \;=\; \sum_{j=1}^{N} (\alpha_{[j]} \mathbb{N}_{U_{[j]}}) \;<\; 0 \end{array}\right. \qquad \forall \alpha \in \boldsymbol{\alpha} \tag{45}$$

where the set $\mathcal{P}(\alpha)$ is a set of m HPD matrices $P_k(\alpha)$ defined by (35). From (44), it can be deduced that:

$$\left\{\begin{array}{l} \mathbb{S}^*(\mathbf{A}) \mathbb{M}_U(A(\alpha), \mathcal{P}(\alpha), G_U) \mathbb{S}(\mathbf{A}) \;<\; 0 \\ \\ \mathbb{S}^*(E) \mathbb{N}_U(A(\alpha), G_U) \mathbb{S}(E) \;<\; 0 \end{array}\right. \qquad \forall \alpha \in \boldsymbol{\alpha} \tag{46}$$

where the matrix function $\mathbb{S}(.)$ is defined in (75). (Note that $\mathbf{A}$ is dependent on α and actually equals $\mathbf{A}(\alpha) = A(\alpha) + E$). Adding both inequalities given in (46) leads to:

$$\mathbb{U}(\mathbf{A}, \mathcal{P}(\alpha)) \;<\; m\mathbb{I}_d \otimes ((E^* E) - \gamma \mathbb{I}_n) \quad \forall \alpha \in \boldsymbol{\alpha} \tag{47}$$

Hence, $\mathbb{U}(\mathbf{A}, \mathcal{P}(\alpha))$ is negative definite (equivalently, $\mathbf{A}$ is $\mathcal{D}_U$-stable) if $E^* E \;\leq\; \gamma \mathbb{I}_n$. With γ reaching its optimal value $\gamma^\star$, condition (42) is recovered. □

Many softwares solving $\mathcal{LMI}$ problems can be exploited, for reasonable values of N, d, and n, to compute the robust $\mathcal{D}_U$-stability bound $\rho^\star$.
In the sequel, we denote by $\rho^{\star^\bullet}$, the bound value obtained by solving (43-44) while imposing (37).

5 Numerical illustrations

In this section, we consider various examples: some of them, quite simple, are borrowed from the literature in order to compare conservatism with other techniques and to emphasize the tractability of our conditions; others, more sophisticated, are here to highlight interest of $\mathcal{D}_U$-stability. The last one is particularly detailed for a better comprehension. Computations were performed on a PC Pentium 266Mhz. The minimization of γ presented in problems (40-41) and (43-44) is achieved owing to LMIToolbox of Matlab [40].

5.1 Robust $\mathcal{D}_R$-stability against additive unstructured uncertainty

In the following examples, the uncertain matrix reads formulation (21) but A is a precisely known.

5.1.1 Example 1

This example may be the first one in the literature related to robustness bounds. It was first introduced in [21] and illustrated many contributions. Thus, the conservatism induced by our bound can be compared to existing results. The considered nominal matrix is:

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \tag{48}$$

Only Hurwitz stability is analysed so $\mathcal{D}_R$ is $\mathbb{C}^-$. The best robustness bound value, to our knowledge was $\rho' = 0.4648$ [36]. Applying our $\mathcal{LMI}$ condition yields $\rho^\star = 0.5303$ (`computation time: 0.11s`).

To appreciate the conservatism of our bounds, we have achieved a function which generates 2000 random instances of the uncertain matrix (possibly including both polytopic and unstructured parts) and which plots, for each instance, the corresponding spectrum. For this example, our function leads to figure 1 where we can see the pole migration. The generated random additive matrices E are of course such that $||E||_2 \leq \rho^\star$. This uncertain root-locus is plotted for every example in this section.

5.1.2 Example 2

This is also a famous example of the literature [27]:

$$A = \begin{bmatrix} -4.3 & -0.4 \\ 0.2 & -3.4 \end{bmatrix} \tag{49}$$

The $\mathcal{EMI}$-region is the disk centred on $(-4; 0)$ with radius 1. Once again, the purpose is to compare the pessimism of our bound with existing results. The bound of [28] is $\rho' = 0.3865$ while we find $\rho^\star = 0.4100$ (`computation time: 0.16s`) which is not too conservative (see figure 2).

5.2 Robust $\mathcal{D}_U$-stability against additive unstructured uncertainty

The numerical illustration is here extended to the case of an $\mathcal{EEMI}$-region. Matrix A is still precisely known. This example, corresponding to an airplane, is used in [36, page 141]. The open-loop model is:

$$A_0 = \begin{bmatrix} -0.1382 & 0.1191 & -0.9747 & 0.1566 \\ -2.0034 & 0.9564 & 0.7325 & 0.0055 \\ 0.3950 & -0.1236 & -0.2371 & -0.0075 \\ 0.0000 & 1.0000 & 0.0760 & 0.0000 \end{bmatrix} \quad ; \quad B_0 = \begin{bmatrix} -0.0082 & 0.3010 \\ -0.5789 & 0.2824 \\ -0.0225 & -0.4184 \\ 0.0000 & 0.0000 \end{bmatrix} \tag{50}$$

The state feedback matrix

$$K = \begin{bmatrix} 5.2678 & 18.7107 & 12.7321 & 60.1235 \\ -43.9951 & -5.6842 & -8.0060 & -38.8788 \end{bmatrix} \tag{51}$$

is computed so that the spectrum of $A = A_0 + B_0 K$ is $\{-10; -6.5; -5.5; -2\}$. The chosen $\mathcal{EEMI}$-region $\mathcal{D}_U$ is the union of three disks respectively centred on $(-10; 0)$, $(-6; 0)$ and $(-2; 0)$, and with respective radii 1, 2 and 1. Owing to that region, we cluster a fast dynamic, an average one and a slow one (separatly) while ensuring a minimum damping ratio $\zeta \simeq 0.8$. Note that such a union of three subregions must be considered as very sophisticated for root-clustering analysis. Applying our condition gives $\rho^\star = 0.0310$ in `25.16s` which is not very conservative in spite of the sophisticated region (figure 3). Besides, only `8.24s` are required when imposing (37) because the number of decision variables reduces from 223 to 127 and the result is $\rho^{\star\bullet} = 0.0308 \simeq \rho^\star$.

5.3 Robust $\mathcal{D}_R$-stability against both uncertainties

Matrix A now belongs to a polytope as defined in (22) so an implicite computation of a parameter-dependent Lyapunov function is performed. For this example, extracted from [41], the $\mathcal{EMI}$-region $\mathcal{D}_R$ is the left open symmetric sector having its corner at the origin and making an angle of $\frac{\pi}{4}$ radians with the imaginary axis. Such a region guarantees a minimum damping ratio of $\sqrt{2}/2$. The model is of the form (24) with $h = 2$ and:

$$A = \begin{bmatrix} -6 & -2 \\ 5 & 0 \end{bmatrix} \quad ; \quad E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \quad ; \quad E_2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \tag{52}$$

In [41], it is proved that $\mathbf{A}$ is robustly $\mathcal{D}_R$-stable if $|\delta_l| \leq 1.0899$, what defines a polytope of matrices with 4 vertices. Here, we show that not only this polytope is robustly $\mathcal{D}_R$-stable but that an unstructured uncertainty can also be accepted since, in `1.54s`, the bound value $\rho^\star = 0.0577$ is reached. For such possible uncertainties, the root-locii are given by figure 4.

5.4 Robust $\mathcal{D}_U$-stability against both uncertainties

5.4.1 Example1

The model is in fact the one given by (50) and (51). However, we now consider that A, the closed-loop matrix, is of the form (24) with $h = 3$ and:

$$E_1 = \begin{bmatrix} 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad E_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix} \tag{53}$$

The chosen region $\mathcal{D}_U$ is still the union of three disks previously mentioned. Considering a polytope defined by the constraint $|\delta_l| \leq 0.1\, \forall l \in \{1; 2; 3\}$, a robust $\mathcal{D}_U$-stability bound is computed in `44min 12s` and equals $\rho^\star = 0.0226$. Some possible pole migrations coresponding to this uncertainty domain are shown on figure 5.
Assuming that constraint (37) is added, the bound is nearly the same *i.e.* $\rho^{\star\bullet} = 0.0222 \simeq \rho^\star$. However, the computation time is much smaller (`3min 13s`) because of the number of decision variables reduces from 433 to 337.

5.4.2 Example 2

In [42], a satellite is modelled by two masses connected by a spring with torque Γ and viscous damping δ:

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -10\Gamma & -10\delta & 10\Gamma & 10\delta \\ 0 & 0 & 0 & 1 \\ \Gamma & \delta & -\Gamma & -\delta \end{bmatrix} ; \; B_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{54}$$

The torque is here assumed constant: $\Gamma = 0.35$. Viscous damping δ is assumed to lie in range $0.1\sqrt{\Gamma/10} \leq \delta \leq 0.2\sqrt{\Gamma/10}$ with $\delta_0 = 0.029$ as a nominal value. Assigning arbitrary spectrum $\{-1.95 \pm 2\mathbf{i}; -2 \pm 2.05\mathbf{i}\}$ by state feedback

$$K = \begin{bmatrix} 4.6721 & -8.9610 & -22.9578 & -7.5810 \end{bmatrix} \tag{55}$$

induces a damping ratio $\zeta \simeq 0.7$. When δ lies in its range, closed-loop state matrix $A = A_0 + B_0K$ is somewhere in a polytope with two vertices (one for each extreme value of δ). Also assume that A is subject to an additive uncertainty E corresponding to some neglected dynamics and a bad precision while implementing K for example. We expect the poles not to be very sensitive with respect to both polytopic and unstructured uncertainties and the damping ratio to lie in about $0.4 \leq \zeta \leq 0.9$. To achieve such an analysis, we choose the union of two disks respectively centred on $-2+2\mathbf{i}$ and $-2-2\mathbf{i}$ and both of radius 1. Such a region $\mathcal{D}_U$ enables ζ to vary in $[0.4114; 0.9114]$. The corresponding matrices R_k are:

$$R_1 = \begin{bmatrix} 7 & 2-2\mathbf{i} \\ 2+2\mathbf{i} & 1 \end{bmatrix} \quad ; \quad R_2 = \begin{bmatrix} 7 & 2+2\mathbf{i} \\ 2-2\mathbf{i} & 1 \end{bmatrix} \tag{56}$$

Note that in this case, the involved matrices are complex thus the problem of computing $\rho^\star$ is harder from a numerical point of view because real and imaginary parts have to be separated [40] and the number of decision variables is 257 (including the conversion from a complex problem to a real one). After `11min 9s`, some HPD matrices $P_{[j]_k} \in \mathbb{C}^{4\times 4}$, $k = 1,2$, $j = 1,2$, and a matrix $G_U \in \mathbb{C}^{12\times 8}$ are computed and yield a bound value $\rho^\star = 0.0142$. For such a polytope and for $||E||_2 \leq \rho^\star$, the uncertain root-locus is given by figure 6. Considering constraint (37) enables to reduce the computation time to `7min 6s` and leads to $\rho^{\star\bullet} = 0.0138$.

6 Conclusion

In this paper, a new method to compute robustness bounds against unstructured uncertainty has been proposed. It enables a large choice of clustering regions including many convex regions and unions of those convex regions. Besides of the unstructured additive uncertainty, another convex polytopic one can be taken into account. In this case, the bound is obtained through the implicit computation of a parameter-dependent Lyapunov function which allows a strong reduction of conservatism. No expression of bound is proposed since the bound value is reached by maximizing a linear criterion under $\mathcal{LMI}$ constraints. Hence, powerful numerical tools can be used for computation and the proposed technique turns to be not only efficient but also tractable.
Some further investigation could concern the extension of the results in a design context. It seems very difficult to use this analytical tool for a robust synthesis purpose because of the various matrices $P_{[j]_k}$. However, some robust pole placement might be computed following the techniques proposed in [43, 36].

Appendix

Proof of theorem 9: some arguments used here can be found in [34, 19].
Sufficiency: assume that (18) holds for some set $\mathcal{P}$. For each eigenvalue λ_i of A, one can find a non-zero vector $v_i \in \mathbb{C}^n$ such that $Av_i = \lambda_i v_i$ and $v_i^* A^* = \lambda_i^* v_i^*$ $\forall i \in \{1,..,n\}$ and from (18), it comes:

$$(\mathbb{I}_d \otimes v_i^*)\mathbb{U}(A,\mathcal{P})(\mathbb{I}_d \otimes v_i) < 0 \quad \forall i \in \{1,...,n\} \tag{57}$$

$$\Leftrightarrow \begin{bmatrix} \mathbb{I}_d \otimes v_i^* & \mathbb{1}_{1,m} \otimes \mathbb{I}_d \otimes \lambda_i^* v_i^* \end{bmatrix} \mathfrak{U}(\mathcal{P}) \begin{bmatrix} \mathbb{I}_d \otimes v_i \\ \mathbb{1}_{m,1} \otimes \mathbb{I}_d \otimes \lambda_i v_i \end{bmatrix} < 0 \quad \forall i \in \{1,...,n\} \tag{58}$$

$$\Leftrightarrow \begin{bmatrix} \mathbb{I}_d^* & (\mathbb{1}_{1,m} \otimes \mathbb{I}_d)\lambda_i^* \end{bmatrix} (\mathbb{I}_d \otimes v_i^*)\mathfrak{U}(\mathcal{P})(\mathbb{I}_d \otimes v_i) \begin{bmatrix} \mathbb{I}_d \\ (\mathbb{1}_{m,1} \otimes \mathbb{I}_d)\lambda_i \end{bmatrix} < 0 \quad \forall i \in \{1,...,n\} \tag{59}$$

$$\Leftrightarrow \begin{bmatrix} \mathbb{I}_d^* & (\mathbb{1}_{1,m} \otimes \mathbb{I}_d)\lambda_i^* \end{bmatrix} U(\underline{w}_i) \begin{bmatrix} \mathbb{I}_d \\ (\mathbb{1}_{m,1} \otimes \mathbb{I}_d)\lambda_i \end{bmatrix} < 0 \quad \forall i \in \{1,...,n\} \tag{60}$$

with $\underline{w}_i$ being a vector of m real positive entries w_{i_k} defined by:

$$w_{i_k} = v_i^* P_k v_i > 0 \quad \forall \{i;k\} \in \{\{1,...,n\} \times \{1,...,m\}\} \tag{61}$$

since the matrices P_k are positive definite.
Hence, (60) means that there exists a suitable vector $\underline{w}_i$ of strictly positive entries such that:

$$f_{\mathcal{D}_U}(\underline{w}_i, \lambda_i) < 0 \quad \forall i \in \{1,...,n\} \Leftrightarrow \lambda_i \in \mathcal{D}_U \quad \forall i \in \{1,...,n\} \quad \square \tag{62}$$

Necessity: it is more discerning to use the expression of $f_{\mathcal{D}_U}(\underline{w}, z)$ given in (17) in order to properly understand this part of the proof. Assume that (62) holds. Then:

$$\forall i \in \{1,...,n\}, \exists \underline{w}_i = \begin{bmatrix} w_{i_1} \\ \vdots \\ w_{i_m} \end{bmatrix} \in \{\mathbb{R}^+/0\}^m \,|\, f_{\mathcal{D}_U}(\underline{w}_i, \lambda_i) < 0 \tag{63}$$

$$\Leftrightarrow \quad \underset{i=1,\dots,n}{\text{diag}} \{f_{\mathcal{D}_U}(\underline{w}_i, z)\} < 0 \tag{64}$$

which, considering (17), can be written

$$\underset{i=1,\dots,n}{\text{diag}} \{\sum_{k=1}^{m}(w_{i_k}(R_{k_{00}} + R_{k_{10}} z + R_{k_{10}}^* \tilde{z} + R_{k_{11}} z\tilde{z}))\} < 0 \tag{65}$$

At this stage, notice this property of the Kronecker product [44]:

$$\forall L_k = \quad \exists S \in \mathbb{C}^{nd \times nd} \text{ (permutation matrix) } | \underset{i=1,\dots,n}{\text{diag}} \{l_{i_k}\} \in \mathbb{C}^{n \times n}, \forall X_k \in \mathbb{C}^{d \times d}, \; k = 1,...,m,$$

$$\sum_{k=1}^{m}(X_k \otimes L_k) = S^*(\sum_{k=1}^{m} \underset{i=1,\dots,n}{\text{diag}} \{l_{i_k} X_k\})S \tag{66}$$

By premultiplying inequality (65) by a suitable S^* and postmultiplying it by S, it comes:

$$\mathbb{U}(\Lambda, \mathcal{W}) < 0 \quad \text{with} \quad \Lambda = \underset{i=1,\dots,n}{\text{diag}} \{\lambda_i\} \tag{67}$$

and where $\mathcal{W}$ is a set of m diagonal matrices defined by:

$$W_k = \underset{i=1,\dots,n}{\text{diag}} \{w_{i_k}\} \quad \forall k \in \{1,...,m\} \tag{68}$$

At this stage, two cases have to be considered :
• If A is diagonalisable, *i.e.* if there exists a matrix $V \in \mathbb{C}^{n \times n}$ such that $\Lambda = V^{-1}AV$, then (67) can be written as follows:

$$\mathbb{U}(V^{-1}AV, \mathcal{W}) < 0 \tag{69}$$

Pre and postmultiplying (69) respectively by $\mathbb{I}_d \otimes V^{-*}$ and $\mathbb{I}_d \otimes V^{-1}$ yields:

$$\mathbb{U}(A, \{V^{-*} W_k V^{-1}\}) < 0 \tag{70}$$

with $\{V^{-*}W_kV^{-1}\}$ denoting the set of m matrices $(V^{-*}W_kV^{-1})$, $k = 1, ..., m$ (this notation is used further).
Inequality (70) is equivalent to (20) (then to (18)) by choosing $P_k = V^{-*}W_kV^{-1} = P_k^* > 0 \quad \forall k \in \{1, ..., m\}$.

• If A is not similar to Λ, one can find a sequence of nonsingular matrices T_l such that:

$$\lim_{l\to\infty}(T_l^{-1}JT_l) = \Lambda \tag{71}$$

where J is the Jordan canonical form of A such that $J = V^{-1}AV$.
Since $\mathbb{U}$ is a continuous function of its first agument, then it follows from (67) that:

$$\lim_{l\to\infty}\mathbb{U}(T_l^{-1}JT_l, \mathcal{W}) < 0 \tag{72}$$

and $\mathbb{U}(T_l^{-1}JT_l, \mathcal{W})$ is negative definite when l is sufficiently large. For such an instance of l, denote T_l by T. It comes:

$$\mathbb{U}(T^{-1}JT, \mathcal{W}) < 0 \tag{73}$$

Pre and postmultiplying (73) respectively by $\mathbb{I}_d \otimes (V^{-*}T^{-*})$ and $\mathbb{I}_d \otimes (T^{-1}V^{-1})$ yields:

$$\mathbb{U}(V^{-*}JV^{-1}, \{V^{-*}T^{-*}W_kT^{-1}V^{-1}\}) = \mathbb{U}(A, \{V^{-*}T^{-*}W_kT^{-1}V^{-1}\}) < 0 \tag{74}$$

which is equivalent to (20) (then to (18)) with the choice $P_k = V^{-*}T^{-*}W_kT^{-1}V^{-1} = P_k^* > 0.\square$

Proof of theorem 14:

Sufficiency: It could also be seen as a special case of matrix elimination lemma. Neverthless, we give an alternative proof. Let the matrix $\mathbb{S}(A)$ be defined by:

$$\mathbb{S}(A) = \begin{bmatrix} \mathbb{I}_{dn} \\ \mathbb{Z}(A) \end{bmatrix} \tag{75}$$

with:

$$\mathbb{Z}(A) = \mathbb{1}_{m,1} \otimes \mathbb{I}_d \otimes A \tag{76}$$

Then, from (32), it comes:

$$\mathbb{S}^*(A)\overline{\mathbb{U}}(A, \mathcal{P}, G_U)\mathbb{S}(A) = \mathbb{U}(A, \mathcal{P}) < 0 \tag{77}$$

which proves sufficiency by virtue of theorem 9. $\square$

Necessity: if A is $\mathcal{D}_U$-stable, then there exist some set $\mathcal{P}$ such that:

$$\mathbb{U}(A, \mathcal{P}) = \mathbb{S}^*(A)\mathfrak{U}(\mathcal{P})\mathbb{S}(A) < 0 \tag{78}$$

One can always find an arbitrary positive scalar number ε that is small enough to check:

$$\mathbb{S}^*(A)\mathfrak{U}(\mathcal{P})\mathbb{S}(A) + \varepsilon\sum_{k=1}^{m}(\mathbb{Z}^*(A)(\underset{k=1,\dots,m}{\operatorname{diag}}\{\mathbb{I}_d \otimes P_k\})\mathbb{Z}(A)) < 0 \tag{79}$$

Respectively define m matrices $\hat{R}_{k_{11}}$ and matrix $\check{\mathfrak{U}}(\mathcal{P})$ by:

$$\hat{R}_{k_{11}} = R_{k_{11}} + \varepsilon \mathbb{I}_d \quad \forall k \in \{1,...,m\} \quad \text{and} \quad \check{\mathbb{U}}(\mathcal{P}) = \begin{bmatrix} \sum_{k=1}^{m} (R_{k_{00}} \otimes P_k) & [R_{k_{10}} \otimes P_k]_{1\le k\le m} \\ ([R_{k_{10}} \otimes P_k]_{1\le k\le m})^* & \mathbb{O}_{dn} \end{bmatrix} \tag{80}$$

Then, inequality (79) is equivalent to:

$$\mathbb{S}^*(A)\check{\mathbb{U}}(\mathcal{P})\mathbb{S}(A) + \sum_{k=1}^{m} (\mathbb{Z}(A)^*(\operatorname*{diag}_{k=1,...,m} \{\hat{R}_{k_{11}} \otimes P_k\})\mathbb{Z}(A)) < 0 \tag{81}$$

which by Schur complement argument, yields:

$$\begin{bmatrix} \mathbb{S}^*(A)\check{\mathbb{U}}(\mathcal{P})\mathbb{S}(A) & \mathbb{Z}^*(A)\left(\operatorname*{diag}_{k=1,...,m} \{\hat{R}_{k_{11}} \otimes P_k\} \right) \\ \left(\operatorname*{diag}_{k=1,...,m} \{\hat{R}_{k_{11}} \otimes P_k\} \right)\mathbb{Z}(A) & -\operatorname*{diag}_{k=1,...,m} \{\hat{R}_{k_{11}} \otimes P_k\} \end{bmatrix} < 0 \tag{82}$$

$$\Leftrightarrow \begin{bmatrix} \sum_{k=1}^{m}(R_{k_{00}}) + \mathbb{S}^*(A) \begin{bmatrix} \mathbb{O}_{dn} & [R_{k_{10}} \otimes P_k]_{1\le k\le m} \\ ([R_{k_{10}} \otimes P_k]_{1\le k\le m})^* & \mathbb{O}_{mdn} \end{bmatrix} \mathbb{S}(A) & \mathbb{Z}^*(A)\left(\operatorname*{diag}_{k=1,...,m} \{\hat{R}_{k_{11}} \otimes P_k\} \right) \\ \left(\operatorname*{diag}_{k=1,...,m} \{\hat{R}_{k_{11}} \otimes P_k\} \right)\mathbb{Z}(A) & \operatorname*{diag}_{k=1,...,m} \{(R_{k_{11}} - \hat{R}_{k_{11}} - \hat{R}_{k_{11}} + \varepsilon \mathbb{I}_d) \otimes P_k\} \end{bmatrix} < 0 \tag{83}$$

$$\Leftrightarrow \mathbb{U}(\mathcal{P}) + \begin{bmatrix} \mathbb{Z}^*(A) \\ -\mathbb{I}_{mdn} \end{bmatrix} \begin{bmatrix} ([R_{k_{10}} \otimes P_k]_{1\le k\le m})^* & \operatorname*{diag}_{k=1,...,m} \{\hat{R}_{k_{11}} \otimes P_k\} \end{bmatrix} +$$

$$\begin{bmatrix} [R_{k_{10}} \otimes P_k]_{1\le k\le m} \\ \operatorname*{diag}_{k=1,...,m} \{\hat{R}_{k_{11}} \otimes P_k\} \end{bmatrix} \begin{bmatrix} \mathbb{Z}(A) & -\mathbb{I}_{mdn} \end{bmatrix} < -\varepsilon \begin{bmatrix} \mathbb{O}_{dn} & \mathbb{O}_{dn,mdn} \\ \mathbb{O}_{mdn,dn} & \operatorname*{diag}_{k=1,...,m} \{\mathbb{I}_d \otimes P_k\} \end{bmatrix} \le 0 \tag{84}$$

The previous inequality implies:

$$\overline{\mathbb{U}}(A, \mathcal{P}, G_U) < 0 \tag{85}$$

with the choice:

$$G_U = \begin{bmatrix} [R_{k_{10}} \otimes P_k]_{1\le k\le m} \\ \operatorname*{diag}_{k=1,...,m} \{\hat{R}_{k_{11}} \otimes P_k\} \end{bmatrix} \quad \square \tag{86}$$

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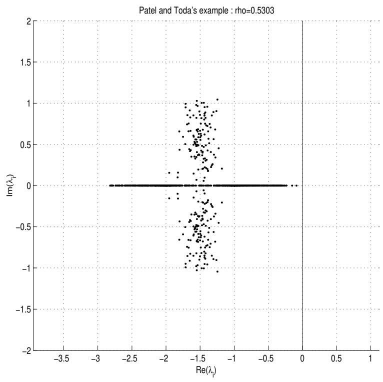

Figure 1: Patel and Toda's example with $\rho^\star = 0.5303$

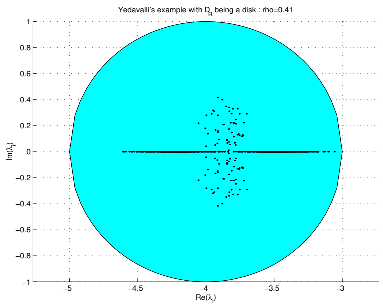

Figure 2: Yedavalli's example with $\mathcal{D}_R$ being a disk and $\rho^\star = 0.4100$

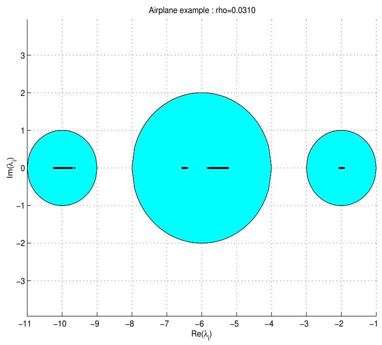

Figure 3: Airplane closed-loop model with $\rho^\star = 0.0310$

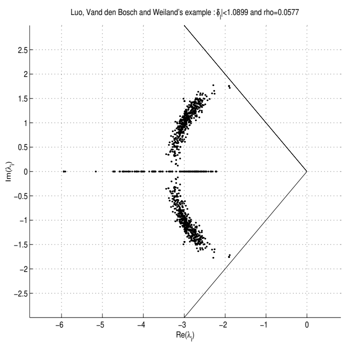

Figure 4: Luo, Van den Bosch and Weiland's example with $|\delta_l| \leq 1.0899$ and $\rho^\star = 0.0577$

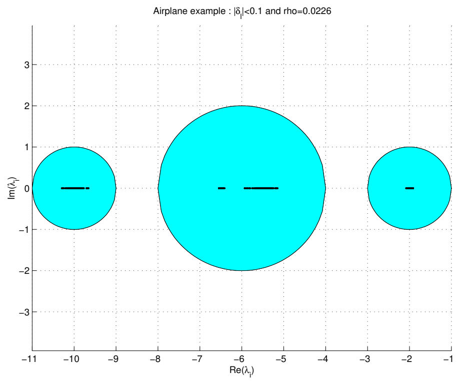

Figure 5: Airplane closed-loop model with $|\delta_l| \leq 0.1$ and $\rho^\star = 0.0226$

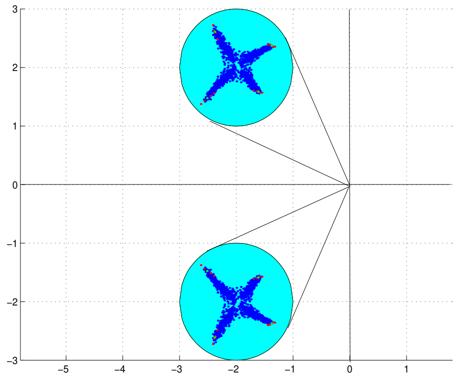

Figure 6: Satellite closed-loop model with $0.0187 \leq |\delta| \leq 0.0374$ and $||E||_2 \leq 0.0142$