Uncertainty domain quantification for continuous-time state-space model

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abstract—In this paper, a new method is introduced to determine the uncertainty domains for multi-input multi-output continuous-time systems. This method uses a particular continuous-time subspace-based algorithm to identify the parameters system. This identification method is specific because the estimated state-space realization is described with the help of a canonical state-space coordinate basis. Then, a bounded error approach is considered to characterize the uncertainty domains of the state-space matrices parameters. A simulation example is included to demonstrate the efficiency of the developed method.

keywords— Continuous-time system identification, multi-input/multi-output systems, subspace-based methods, uncertainty domain, bounded error approach.

I. INTRODUCTION

In many applications (modeling, diagnosis, control), it is interesting to deal with continuous-time (CT) model in order to have access to the physical parameters of the system. This choice can be easily justified by the fact that most of the physical phenomena are viewed in CT setting. In recent years, the CT identification problem has received a wide attention [1], [2]. In the dedicated literature, two main groups of CT identification methods can be distinguished: the indirect and the direct methods. The first family consists in determining a discrete-time (DT) model via conventional identification algorithms [3], then converting this representation into a CT model. Unfortunately, the methods gathered into this family suffer from many limitations, as [4], [5]

− the difficult choice of the sampling period,
− the use of the matrix logarithm which may produce complex arithmetic when the matrices have negative eigenvalues,
− the tricky translation of zeros in the CT domain [6].

All these restrictions can be alleviated with the direct approach which consists in estimating directly a CT model from sampled I/O data [4], [5]. Two different approaches can also be considered. The first one is based on the definition of linear parameter models and the use of least-squares algorithms. The main problem of these techniques is the computation of the successive I/O time derivatives. The solution often rests on the use of linear filters, integral methods or modulating functions (see [2] for a recent overview and discussions about this problem). The second one consists in using non-linear optimization algorithms, as developed in the output error framework [3]. Interesting from a theoretical point of view, these techniques often lead to local minima problems and require good initial models to ensure global convergence.

When the system has complex dynamics, the estimated model is only an approximation of the real process. To deal with this limitation, an estimate of the model accuracy is required when a reliable design is looked for. For this purpose, the identification procedure must deliver not only an accurate nominal model, but also a reliable estimate of the uncertainty associated with the model.

In system identification theory, two main philosophies are adopted to characterize the uncertainty domains. The first one is based on statistical assumptions [7], [3]. The techniques sharing this basic idea assume that the disturbances acting on the system are realizations of random variables [8], [9]. Because the information on the measurement noises is not often available and difficult to verify [10], a different characterization way can be considered [11]. This approach is mainly based on deterministic hypotheses, e.g., on the assumption that the residuals are unknown-but-bounded [12], [13]. This basic idea has given rise to a number of techniques usually addressed as bounded error methods or set membership identification [14], [15], [16], [17]. The main drawback of this approach is its dependence on the way the bound is determined. Notice indeed that the error comes from two sources (the unmodeled dynamics and the noise affecting the data) which makes the bound determination quite tricky in practice.

In order to deal with single-input single-output (SISO) and multi-input multi-output (MIMO) systems in a similar way, state-space model representations are considered hereafter. To get an accurate nominal model of the system, a specific subspace-based method is introduced in this paper. Contrary to the classic subspace-based algorithms [18], [19], [20], the proposed technique does not give access to a fully-parametrized form but leads to a state-space representation, expressed in a fixed basis, with a minimal number of parameters, even for MIMO systems. Thanks to the fixed and minimal state-space matrices structure provided by the identification algorithm, the model is written into an input-output (I/O) form linear in the parameters (LP) which makes the description of the uncertainty areas easier. These domains are derived from the analysis of particular quadratic criteria. The final objective is to get realistic uncertainty domains that contain all kinds of stochastic disturbances. Thus, an unknown-but-bounded approach is considered to reach this goal. More particularly, an easy-tuning method is proposed to fix the value of the required bound and no particular assumption is made on the noise.
The state-space representation

Assumption A. The state-space representation (1) is minimal, i.e., $(A, B)$ is controllable and $(A, C)$ is observable.

Assumption B. $v$ is statistically independent of the input signal $u$.

Assumption C. The system order $n_x$ is assumed to be known a priori.

No particular assumption is made concerning the (statistical or morphological) properties of $v$. From a practical point of view, it is usual to consider a measurement noise constant between two samples [23], [24], i.e., $v(t) = v(kT_s)$, for $kT_s < t < (k + 1)T_s$ where $T_s$ is the sampling period. More precisely, $v(kT_s)$ is a DT zero-mean white noise satisfying

$$
E[v(kT_s)v^\top(t)] = R_v \delta(k, l)
$$

where $\delta(k, l)$ is the Kronecker delta function and $E[\bullet]$ the mathematical expectation [3]. However, from a theoretical point of view (mainly as far as the uncertainty domain description is concerned), it is assumed hereafter that the model residuals, i.e., the difference between the simulated and measured outputs, are bounded.

Considering the aforementioned assumptions, the CT identification problem solved in this paper can be stated as follows: given realizations of $[u(t_k)]_{k=1}^N$ and $[y(t_k)]_{k=1}^N$, with $t_k = kT_s$, of the input and output signals generated by a system of the form (1) on a finite but sufficiently wide time horizon $N$, estimate the matrices $(A, B, C)$ represented in a particular state-space form and characterize the estimated parameters quality.

To reach this goal, two linked problems are solved. Firstly, the system matrices (1) are identified using a specific CT subspace-based algorithm (see Section III) because that the estimated state-space matrices are represented directly in a minimally-parametrized form, i.e., with a number of parameters equal to $n_x(n_u + n_y)$. Then, the estimated parameters uncertainty domains are described through the analysis of particular iso-level curves and a bounded error approach (see Section IV).

III. STATE-SPACE MATRICES ESTIMATION USING A CONTINUOUS-TIME IDENTIFICATION METHOD

A. The propagator method: a reminder

The key problem of this identification method, named the propagator method, is the consistent estimation, from measured I/O data samples, of the extended observability matrix column space defined as

$$
\Gamma_i = \Gamma_i(A, C) = [(C)\top \quad (CA)\top \quad \cdots \quad (CA^{i-1})\top]\top
$$

where $i$ is a user-defined integer such that $i \geq n_x$. It is indeed relatively straightforward to extract the state-space matrices from $\Gamma_i$ by exploiting some of its particular properties of this matrix. The starting point of the CT propagator method is the following CT data equation [25], [26]

$$
\mathcal{M}\{y_i\} = \Gamma_i\mathcal{M}\{X\} + \mathcal{H}_i\mathcal{M}\{U_i\} + \mathcal{M}\{V_i\}
$$

with $\mathcal{H}_i$ is a Toeplitz matrix composed of the Markov parameters of the system,

$$
\mathcal{M}\{y_i\} = 
\begin{bmatrix}
M_0^y\{y(t_k)\} \\
M_0^y\{y(t_{k+1})\} \\
\vdots \\
M_0^y\{y(t_{N-k})\}
\end{bmatrix}
$$

and

$$
\mathcal{M}\{X\} = 
\begin{bmatrix}
M_0^x\{x(t_k)\} \\
M_0^x\{x(t_{k+1})\} \\
\vdots \\
M_0^x\{x(t_{N-k})\}
\end{bmatrix}
$$

where $\mathcal{M}[\bullet]$ stands for a transformation applied to the sampled data in order to approximate the I/O data time-derivatives [26].

Equation (3) is equivalent to the so-called data equation introduced for the DT subspace-based identification [27]. Assuming that the input is sufficiently exciting (see [28] for conditions), the initial condition are zero and the transformation does not remove any state variable, i.e., rank $[\mathcal{M}\{X\}] = n_x$, it is possible to adapt the DT propagator method [22] to this CT problem. So as not to overload the paper, only the main steps of this technique are described hereafter. For instance, equations are introduced without taking into account the disturbances.

As a subspace-based technique, the first step of the propagator method consists in applying an orthogonal projection of the row space of $\mathcal{M}\{Y_i\}$ onto the complement of the row space of $\mathcal{M}\{U_i\}$

$$
\mathcal{M}\{Y_i\} \Pi_{\mathcal{M}\{U_i\}} = \Gamma_i\mathcal{M}\{X\} \Pi_{\mathcal{M}\{U_i\}}\mathcal{M}\{U_i\}
$$

where

$$
\Pi_{\mathcal{M}\{U_i\}} = I_{n_u} - \mathcal{M}\{U_i\}\mathcal{M}\{U_i\}\top\mathcal{M}\{U_i\}\top^{-1}\mathcal{M}\{U_i\}
$$

This projection can be computed in a stable and efficient way by resorting to the RQ factorisation [29].
in order to remove the forced response and to solve the
unknown matrix \( H_1 \) problem. Then, its second step tries
to estimate the observability subspace. To reach this goal,
a particular similarity transformation is introduced in or-
der to fix the estimated state-space matrices coordinates
basis and to simplify the estimation procedure by using
least-squares algorithms. More precisely, it is based on the
following proposition (see [30, Lemma 1] for a proof)

**Proposition 1.** Assuming that the system is observable
and the state matrix \( A \) is non-derogatory [23], introducing
\( \kappa = [\kappa_1 \; \kappa_2 \; \cdots \; \kappa_{n_y}] \in \mathbb{R}^{1 \times n_y} \) a vector generated ran-
domly from a uniform distribution, it holds with probability
one that \( \text{rank}\{\Gamma_{n_x} (A, \kappa^\top C)\} = n_x \).

This proposition states that, for a vector \( \kappa \) generated ran-
domly from a uniform distribution, the system dynamics
are observable from the following auxiliary output

\[
y_o(t) = \sum_{j=1}^{n_y} \kappa_j y_j(t) = \kappa^\top C x(t)
\]

where \( y_j \) is the \( j \)th system output. In order to introduce
explicitly this auxiliary output in the I/O data and use,
afterwards, \( \Gamma_{n_x} (A, \kappa^\top C) \) as a similarity transformation,
let us define a matrix

\[
K = \begin{bmatrix}
\kappa_1 & \kappa_2 & \cdots & \kappa_{n_y} \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix} \in \mathbb{R}^{n_y \times n_y}.
\]

Then, it is easy to verify that \( y_o \) substitutes\(^4\) for the first
system output in \( \hat{y} = K y \). Furthermore, using this trans-
formation and this new output signal \( \hat{y} \), Eq. (6) becomes

\[
\mathcal{M} \{ Y_i \} \Pi_{\mathcal{M}(U_i)}^\top = \Gamma_{i} \mathcal{M} \{ X \} \Pi_{\mathcal{M}(U_i)}^\top
\]

with

\[
\begin{align}
\Gamma_i &= \Gamma_i (A, \hat{C}) \\
\hat{C} &= \begin{bmatrix}
\hat{c}_1 \\
\hat{c}_2 \\
\vdots \\
\hat{c}_{n_y}
\end{bmatrix} = \begin{bmatrix}
\sum_{j=1}^{n_y} \kappa_j c_j \\
c_2 \\
\vdots \\
c_{n_y}
\end{bmatrix}
\end{align}
\]

where \( c_j, j \in [1, n_y] \), are the rows of \( C \). Noting that \( \Gamma_i \)
has full column rank, a permutation matrix \( S \) can be in-
cluded in order to reorder the rows of \( \Gamma_i \), and to ensure
that its first \( n_x \) rows are linearly independent, i.e.,

\[
S \Gamma_i = \begin{bmatrix}
\Gamma_{n_x} (A, \hat{c}_1) \\
\Psi
\end{bmatrix}.
\]

By construction, \( \Gamma_{n_x} (A, \hat{c}_1) \) has full rank \( n_x \). Thus, \( \Psi \)
can be written as a linear combination of \( \Gamma_{n_x} (A, \hat{c}_1) \) or,
similarly, there is a unique operator \( P \in \mathbb{R}^{(n_y f - n_x) \times n_x} \),
named the propagator [21], such that

\[
\Psi = \Pi \Gamma_{n_x} (A, \hat{c}_1).
\]

Furthermore, using \( \Gamma_{n_x} (A, \hat{c}_1) \) as a similarity trans-
formation, it is easy to see that

\[
\mathcal{M} \{ Y_i \} \Pi_{\mathcal{M}(U_i)}^\top = \begin{bmatrix}
I_{n_y} \\
\Gamma_{n_x} (A, \hat{c}_1) \mathcal{M} \{ X \} \Pi_{\mathcal{M}(U_i)}^\top
\end{bmatrix}
\]

This relation shows that the observability subspace can be
described in a particular basis if the propagator is esti-
bated beforehand from the I/O signals. Now, many effi-
cient algorithms are available to estimate the propagator
consistently, even when colored disturbances act on the sys-
tem (see [31], [22], [32] for some details in the DT frame-
work). Most of these algorithms are based on the following
cost function

\[
\| Z_2 - P Z_1 \|_F^2
\]

with \( Z_1 \) and \( Z_2 \) are the components of \( \mathcal{M} \{ Y_i \} \Pi_{\mathcal{M}(U_i)}^\top \)
corresponding respectively to the identify matrix \( n_x \) rows
and to the \( n_y f - n_x \) rows of \( P \).

As soon as \( P \) is estimated accurately, it is easy to show
that\(^5\)

\[
\mathcal{A} = \begin{bmatrix}
I_{n_y} \\
0
\end{bmatrix} (2 : n_x + 1, :) = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

\[
c_j = \begin{cases}
1 & 0 & \cdots & 0 \\
(j - 1) i + 1, : & \text{for } j \in [2, n_y]
\end{cases}
\]

with \( a_j, j \in [0, n_x - 1] \), the coefficients of the character-
istic polynomial of \( A \) [32]. These equations clearly show
that the \( \mathcal{A} \) and \( C \) parameters are obtained directly from \( P \)
by selecting particular rows of this least-squares estimated
matrix. Once \( \mathcal{A} \) and \( C \) are available, the matrix \( B \) can be
obtained by solving a the following linear problem [20]

\[
\hat{B} = \arg \min_{\mathcal{B}} \mathcal{B} \left( \mathcal{Y}(t) - \left( \int_0^t \mathcal{C} U(\tau) \otimes e^{\mathcal{A}(t-\tau)} d\tau \right) \mathcal{C}(t) \right) \vec{C}(\mathcal{B})
\]

where \( \otimes \) stands for the Kronecker product and vec(\( \bullet \)) the
vectorization operator [33].

**IV. UNCERTAINTY DOMAIN DETERMINATION**

The identification procedure introduced beforehand has
several interesting properties as far the uncertainty do-
main description problem is concerned. First of all, by
using least-squares algorithms, the classic tools developed
for linear regression can be easily adapted in this MIMO
state-space framework. Furthermore, contrary to the stan-
dard CT SMI methods, the estimated state-space matrices
are minimally parametrized and expressed into a canonical
state-space coordinates basis. Thus, the problem of basis
reproducibility inherent in SMI is circumvented and the
uncertainty domain computational load is cut down dras-
tically.

\(^4\)The same procedure can be applied for any output \( y_i, i \in [1, n_y]\).

\(^5\)Italic bold letters are used to stress the fact that these matrices
are related to a particular state-space coordinates basis.
In this section, a bounded error approach is presented to determine the uncertainty domain. This method assumes that the residuals, i.e., $\varepsilon = y - \hat{y}$ (with $\hat{y}$ the estimated output), are bounded. A particular ellipsoidal iso-level method proposed in [34], [35] is more particularly adopted. In the following, this bounded error method is firstly introduced in a quite general case, i.e., for any linearly parameterized MIMO system. Then, thanks to the propagator method, this approach is applied to determine firstly the $(A, C)$ parameters uncertainty domains and then the $B$ parameters uncertainty domains.

### A. General case

Consider the following model linear in the parameters

$$y(t) = \Phi^T(t)\theta + v(t)$$  \hspace{1cm} (17)

with $y(t) \in \mathbb{R}^{n_y}$ the output vector, $\theta \in \mathbb{R}^{n_{\theta}}$ the parameters vector and $\Phi^T(t)$ the regressor. In this section our objective is to determine a domain $\mathcal{D}$ such that the true parameters $\theta^0 \in \mathcal{D}$.

In [3], a method to describe the uncertainty domain is developed by using the statistical properties of the prediction error (PE) algorithm. In this technique, the input sequence is assumed to be persistently exciting [3] and the system is stable. Using these assumptions, the PE identification procedure delivers an unbiased identified parameter vector. This vector has the property of being asymptotically a random vector with gaussian distribution, a mean equal to $\theta^0$ and covariance equal to $\lambda(\frac{1}{N}\sum_{t=1}^{N} \Phi^T(t)\Phi(t))^{-1}$ where $\lambda$ is the noise variance. In a different way,

$$\hat{\theta} \in \mathbb{N} \left( \theta^0, \lambda(\frac{1}{N}\sum_{t=1}^{N} \Phi^T(t)\Phi(t))^{-1} \right)$$  \hspace{1cm} (18)

where $\hat{\theta}$ is an arbitrary estimate of $\theta$. Consequently, the uncertainty domain can be given by

$$(\hat{\theta} - \theta^0)^\top(\frac{1}{N}\sum_{t=1}^{N} \Phi^T(t)\Phi(t))(\hat{\theta} - \theta^0) < \alpha \lambda$$  \hspace{1cm} (19)

with $\alpha$ is a user-defined probability level. This is an ellipsoidal domain centered at $\theta^0$ whose shape is given by $\lambda(\frac{1}{N}\sum_{t=1}^{N} \Phi^T(t)\Phi(t))$. This approach assumes that $\hat{\theta}$ is asymptotically a random vector with gaussian distribution and mean value $\theta^0$. Furthermore, the description of this confidence domain depends on the prior knowledge of the noise variance.

Because our objective is to characterize the uncertainty domain without any assumption on the noise and only the information contained in the residuals, the idea presented in [3] is adapted hereafter.

By analyzing Eq. (17), it is obvious that the least-squares estimation of this parameter vector can be obtained by minimizing the following cost function [3, Appendix II]

$$J(\theta) = \frac{1}{N} \sum_{t=1}^{N} (y(t) - \Phi^T(t)\theta)^\top(y(t) - \Phi^T(t)\theta)$$

$$= \frac{1}{N} \sum_{t=1}^{N} \varepsilon^\top(t, \theta)\varepsilon(t, \theta)$$  \hspace{1cm} (20)

where $\varepsilon(t, \theta) = y(t) - \Phi^T(t)\theta$. Then, it is easy to see that

$$J(\theta) = \frac{1}{N} \sum_{t=1}^{N} (y(t) - \Phi^T(t)\hat{\theta}(t))^\top(y(t) - \Phi^T(t)\hat{\theta}(t)) +$$

$$\text{computable residuals}$$

$$\text{ellipsoid centered on } \hat{\theta}$$

When $\theta = \theta^0$, we have

$$(\hat{\theta} - \theta^0)^\top(\frac{1}{N}\sum_{t=1}^{N} \Phi^T(t)\Phi(t))(\hat{\theta} - \theta^0) =$$

$$J(\theta^0) - \frac{1}{N} \sum_{t=1}^{N} (y(t) - \Phi^T(t)\hat{\theta}(t))^\top(y(t) - \Phi^T(t)\hat{\theta}(t)).$$  \hspace{1cm} (21)

By choosing $M^2 = J(\theta^0) - \frac{1}{N} \sum_{t=1}^{N} (y(t) - \Phi^T(t)\hat{\theta}(t))^\top(y(t) - \Phi^T(t)\hat{\theta}(t))$, the real parameter vector is located on the border of the domain $\mathcal{D}$. On the other side, to be sure that $\theta^0 \in \mathcal{D}$ the bound $M$ must be chosen such that

$$M^2 > J(\theta^0) - \frac{1}{N} \sum_{t=1}^{N} (y(t) - \Phi^T(t)\hat{\theta}(t))^\top(y(t) - \Phi^T(t)\hat{\theta}(t)).$$  \hspace{1cm} (22)

In this case, we have

$$J(\theta^0) - \frac{1}{N} \sum_{t=1}^{N} (y(t) - \Phi^T(t)\hat{\theta}(t))^\top(y(t) - \Phi^T(t)\hat{\theta}(t)) < J(\theta^0).$$  \hspace{1cm} (23)

Now,

$$J(\theta^0) = \frac{1}{N} \sum_{t=1}^{N} v(t)^\top v(t).$$  \hspace{1cm} (24)

Thus,

$$J(\theta^0) - \frac{1}{N} \sum_{t=1}^{N} (y(t) - \Phi^T(t)\hat{\theta}(t))^\top(y(t) - \Phi^T(t)\hat{\theta}(t))$$

$$\leq \frac{1}{N} \sum_{t=1}^{N} v(t)^\top v(t).$$

Consequently, we can choose

$$M^2 = \frac{1}{N} \sum_{t=1}^{N} v(t)^\top v(t).$$  \hspace{1cm} (25)

Because it is difficult to have access to $v$, it is suggested to fix the size of the ellipsoid domain using the information contained in the residuals $\varepsilon(\hat{\theta})$. Indeed, $\varepsilon(\hat{\theta})$ can be written as follows

$$\varepsilon(t, \hat{\theta}) = y(t) - \Phi^T(t)\hat{\theta}(t) = v(t) - \Phi^T(t)(\theta^0 - \hat{\theta}).$$  \hspace{1cm} (26)

Let us define $\tilde{M} = \max |\varepsilon(t, \hat{\theta})|$. Then, by construction

$$\tilde{M} \leq \max |v(t)| + \max |\Phi^T(t)(\theta^0 - \hat{\theta})|. $$  \hspace{1cm} (27)
By assuming that \( \hat{\theta} \) is unbiased estimate of \( \theta^0 \) and \( \Phi \) is not correlated with \( v \), we obtain
\[
\mathbb{E} \left\{ \max \left| \Phi^T (t) (\theta^0 - \hat{\theta}) \right| \right\} = \max \left| \Phi^T (t) \right| \mathbb{E} \left\{ (\theta^0 - \hat{\theta}) \right\} = 0.
\]
Thus, the average value of \( \varepsilon(t, \hat{\theta}) \) tends to \( v \). So, \( \hat{M} \) is an unbiased estimate of \( M \) if \( \hat{\theta} \) is an unbiased estimate of \( \theta^0 \). When \( \mathbb{E} \left\{ (\theta^0 - \hat{\theta}) \right\} \neq 0, \hat{M} \) contains the error due to the noise \( v \) as well as the modeling error \( (\theta^0 - \hat{\theta}) \). This property presents an important advantage for the developed uncertainty domain method. Indeed, these regions allow unbiased estimation of \( \theta^0 \) state-space parameters uncertainty domains. 

Consequently, the uncertainty domain can be given by
\[
(\hat{\theta} - \theta^0)^T \left( \frac{1}{N} \sum_{t=1}^{N} \Phi^T (t) \Phi (t) \right) (\hat{\theta} - \theta^0) < \hat{M}^2
\]
with \( \hat{M} = \max \left| \varepsilon(t, \hat{\theta}) \right| \).

Moreover, this method allows the characterization of the parameter vector uncertainty domain using only the information contained on the residuals. So, for each Monte Carlo simulation, we determine an ellipsoidal domain centered in \( \hat{\theta} \) which contains the real parameter \( \theta^0 \).

The approach introduced beforehand assumes that the model is LP. As seen in §III, \( \mathcal{A} \) and \( \mathcal{C} \) are estimated by solving the quadratic cost function (14). More precisely, \( \mathcal{A} \) and \( \mathcal{C} \) parameters are directly related to the \( \mathcal{P} \) estimate as shown in Eq. (15). Because \( \mathcal{P} \) is a least-squares estimate, the problem of the \( (\mathcal{A}, \mathcal{C}) \) estimation can be easily written in a LP form. As far as \( \mathcal{B} \) is concerned, it has been shown that the system output can be expressed linearly with respect to \( \mathcal{B} \) (see Eq. (16)). Similarly, this estimation problem can be written in a LP form. Thus, the approach introduced in §III-A can be adopted to determine all the state-space parameters uncertainty domains.

B. \( \mathcal{A} \) and \( \mathcal{C} \) parameters uncertainty domains

The goal of this subsection is to adapt the approach introduced beforehand to the \( (\mathcal{A}, \mathcal{C}) \) parameters. As shown in §III-A, these parameters are directly extracted from the estimated follower matrix \( \mathcal{P} \). More precisely, these parameters are gathered in the following sub-matrix\(^6\)
\[
\hat{\mathcal{P}} = \begin{bmatrix}
\hat{\mathcal{P}}(n_x + 1, :) \\
\hat{\mathcal{P}}(j - n_y + 1, :)
\end{bmatrix} = \begin{bmatrix}
\mathcal{A}(n_x, :) \\
\mathcal{C}(j, :)
\end{bmatrix}, \quad j \in \{2, n_y\}.
\]

This sub-matrix can be estimated considering the following linear regression
\[
\mathcal{J}_p = \left\| \hat{Z}_2 - \hat{\mathcal{P}} \hat{Z}_1 \right\|_F^2
\]
where \( \hat{Z}_1 \) and \( \hat{Z}_2 \) are respectively deduced from \( \mathcal{Z}_1 \) and \( \mathcal{Z}_2 \) in a compatible way with \( \hat{\mathcal{P}} \). Using the vectorization and the Kronecker product \( \otimes \), it is easy to verify that
\[
\text{vec}(\hat{Z}_2) = (I_{n \times 1} \otimes \hat{Z}_1) \text{vec}(\hat{\mathcal{P}}^T) \otimes \theta_p.
\]

In a similar way as in §IV-A, the uncertainty domain can be given by
\[
(\hat{\theta}_p - \theta^0_p)^T \left( \frac{1}{N} \sum_{t=1}^{N} \Phi^T (t) \Phi (t) \right) (\hat{\theta}_p - \theta^0_p) < \hat{M}^2_p
\]
where \( \theta^0_p \) is the real parameter vector and \( \hat{\theta}_p \) is an arbitrary estimate of \( \theta^0_p \). This equation is equivalent to Eq. (28). Thus, the level of iso-criterion curves can be fixed as \( \hat{M}^2_p = \max \left| \varepsilon_p(t, \hat{\theta}_p) \right| \) with \( \varepsilon_p(t, \hat{\theta}_p) = \mathcal{Z} - \Phi^T (t) \hat{\theta}_p \).

Thus, the \( \mathcal{A} \) and \( \mathcal{C} \) parameters uncertainty domains are an ellipsoid centered at \( \hat{\theta}_p \) whose main directions are given by \( \sum_{t=1}^{N} \Phi^T (t) \Phi (t) \).

C. \( \mathcal{B} \) parameters uncertainty domains

As far as the \( \mathcal{B} \) parameters uncertainty domains are concerned, a similar approach can be considered. Noticing that \( \mathcal{B} \) is estimated using the least-squares cost function
\[
\mathcal{J}_b(\theta_b) = \frac{1}{N} \sum_{t=1}^{N} \varepsilon^T (t, \theta_b) \varepsilon (t, \theta_b)
\]
with \( \varepsilon^T (t, \theta_b) = y(t) - (\mathcal{C} \int_0^t u^T (\tau) \otimes e^{\mathcal{A}(t-\tau)} d\tau) \theta_b \) and \( \theta_b = \text{vec}(\mathcal{B}) \). The main difficulty associated with this criterion is that the estimated \( \mathcal{A} \) and \( \mathcal{C} \) matrices are explicitly involved. Thus, in order to compute the \( \mathcal{B} \) parameters uncertainty domains, it is paramount to take into account the \( \mathcal{A} \) and \( \mathcal{C} \) uncertainties determined in the previous step. By using the first order approximation of the exponential function Taylor series, the approximated cost function satisfies
\[
\mathcal{J}_b(\theta_b) \approx \frac{1}{N} \sum_{t=1}^{N} (b(t) - (\mathcal{C} \int_0^t u^T (\tau) \otimes e^{\mathcal{A}(t-\tau)} d\tau) (\theta_b - \tilde{\theta}_b))^T
\]
\[
\mathcal{J}_b^T (\theta_b) \theta_b
\]
where \( \mathcal{A} \) and \( \mathcal{C} \) are an arbitrary estimate of \( \mathcal{A} \) and \( \mathcal{C} \) estimation and
\[
b(t) = y(t) - \Phi_B (t) \tilde{\theta}_B - (d\mathcal{C} \int_0^t u^T (\tau) \otimes e^{\mathcal{A}(t-\tau)} d\tau) \tilde{\theta}_B
\]
\[-(\mathcal{C} \int_0^t u^T (\tau) \otimes e^{\mathcal{A}(t-\tau)} d\mathcal{A}(t - \tau) d\tau) \tilde{\theta}_B \]
with \( d\mathcal{A} = \mathcal{A} - \hat{\mathcal{A}} \) and \( d\mathcal{C} = \mathcal{C} - \hat{\mathcal{C}} \). Finally,
\[
(\hat{\theta}_p - \theta^0_p)^T \left( \frac{1}{N} \sum_{t=1}^{N} \Phi^T (t) \Phi (t) \right) (\hat{\theta}_p - \theta^0_p) < \hat{M}^2_p.
\]

Following the same steps as in §IV-A, the level \( \hat{M}^2_p \) is equal to \( \hat{M}^2_p = \max \left| \varepsilon_p(t, \hat{\theta}_p) \right| \). So, the \( \mathcal{B} \) parameters uncertainty domains are an ellipsoid centered at \( \hat{\theta}_p \) whose main directions are given by \( \sum_{t=1}^{N} \Phi^T (t) \Phi (t) \).
V. Numerical Example

In order to show the performance of the method described beforehand, the following state-space matrices are used:

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
-3 & -2 & -1 \\
-1 & -2 & -1 \\
\end{bmatrix}
B = \begin{bmatrix}
1 & 1 \\
2 & 1 \\
1 & 2 \\
\end{bmatrix}
C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

The input signal is made up with two independent pseudo random binary sequences of size \(N = 1000\). The sampling period is chosen equal to 0.1s. A Monte Carlo simulation of 200 runs is carried out. A white Gaussian noise is added on both outputs such that the signal to noise ratio (SNR) equals 35dB. The initial state vector is chosen equal to zero. To reduce the noise effect, past inputs are used as instruments variable.

The propagator method is used to identify the system parameters. In this method, the I/O data are filtered using the RPM filter [36], [26]. In order to underline the efficiency of the propagator method, it is proposed to compare the estimated poles obtained by the CT propagator method with the RPM+PI-MOESP method developed in [26]. This algorithm is an extension of the classic DT PI-MOESP algorithm by filtering the I/O data using the RPM filter. In both cases, the past and future horizons are chosen equal to 4.

Fig. 1 shows the estimated poles obtained via the CT propagator method and the RPM+PI-MOESP method. It clearly appears that both methods give quite good poles estimates. Fig. 2 plots the estimated parameters (\(A(3,1)\), \(A(3,2)\)), symbolized by (\(\ast\)) for each Monte Carlo simulation, the real parameters (\(\times\)) and the mean value of the estimated parameters (\(\dagger\)). In some cases, the system parameters are outside of the ellipsoid \(D\) centered in the estimated parameters vector. In such cases, we define the failure rate measure as the percentage of realizations for which the parameters are outside of \(D\). The failure ratio equals 7% for the couple (\(A(3,1), A(3,1)\)). Fig. 3 shows the estimated parameters (\(B(1,1), B(1,2)\)) symbolized by (\(\ast\)), the true parameters (\(\times\)) and the mean value of the estimated parameters (\(\dagger\)). The failure rate is equal to 10%. These two figures show that the proposed method to determine the parameters uncertainty domains gives good results because the failure rate is not very important in both cases.

VI. Conclusion

In this paper, the problem of the uncertainty domain determination for MIMO continuous-time state-space systems is considered. To reach this problem, two linked problem are solved. Firstly, a particular subspace-based method is used to identify the system parameters. This technique has the particularity to estimate the state-space system with a fixed structure and a minimal number of parameters. Then, the state-space matrices parameters uncertainty domains are described using a bounded error approach by adapting particular cost functions. These uncertainty regions allow the quantification of the error due to the noise affecting the system output as well as the modeling error.

The simulation results have emphasized the reliability of the developed method.

References

Fig. 3. Level surfaces of the cost function $J_{\mathcal{G}}$. The real parameters are symbolized by a red cross ($\times$), the estimated parameters by a blue cross ($\ast$) and the mean value of the estimated parameters by a black cross ($+$). Black discs ($\ast$) are finally the failure draws.


